How did you go bankrupt? Two ways. Gradually, then suddenly.

Ernest Hemingway, The Sun Also Rises

A FORMAL INTRODUCTION TO GDP-INDEXED BONDS

Michele Manna
Bank of Italy, Market Operations Directorate
1. Introduction and scope of the work

A debtor is better off if the cost of his debt matches the ups and downs of his economic fortunes: no doubt about it. While it is possible to bear a higher cost of debt in good times when income rises, it definitively helps if the cost falls in downturns. This is true for households, firms and the government. In finance theory, this concept goes under the name of “state-contingent debt”, which in a broad sense can be understood as a contract which foresees a revision in the timing or amount (possibly both) of the payments due by the debtor when pre-defined external circumstances apply.

This is the gist of the proposal put forward by a number of scholars and institutions arguing in favor of the issuance by governments of GDP-indexed bonds (“the GDP-Is”), that is securities whose cash outflows by the Treasury on account of the coupon and/or principal vary with the own GDP dynamics. It is straightforward to understand that such a class of bonds has the potential to cut the cost of servicing the public debt exactly at the point in time when it could be more acutely needed: namely by freeing space in the budget when an unforeseen recession calls for a supportive fiscal policy.

In concept, the GDP-Is are an insurance contract: against a premium, the bondholder takes on the risk of unforeseen changes in GDP through the life of the bond; in turn, the Treasury pays the premium and buys the insurance. Charts 1 and 2, shown at the end of this section, can offer some colour of the issues at stake. Chart 1 plots the actual changes in real GDP for the euro area in 2004-2012 together with the associated one- and two-year advance forecasts by the IMF. The message should be clear: actual changes may deviate, sometimes significantly, from what even first class economists can forecast. Chart 2, a re-print from Bank of England (2016), delivers the results of a simulation of debt-to-GDP ratios over a 20-year horizon under the alternatives of debt financing through nominal (conventional) bonds and GDP-Is. The recourse to the latter class of bonds narrows the range of possible patterns of the ratio, making it less likely that negative growth shocks eventually push this ratio over the roof. However, this comes at a cost (the premium of the insurance contract) which incrementally adds to the debt. This is signaled by the fact that the solid green line, the median scenario when debt is financed through GDP-Is, stands above the corresponding solid yellow line, the median scenario under the conventional bonds.

The idea of bonds indexed to national income is not novel and a first sure reference goes back to Shiller (1993) who popularized the concept of “large international markets for long-term claims on national and occupational incomes, as well as for illiquid assets such as real estate”. However, it is in more recent years, clearly in synch with the severe recession that hit a number of advanced economies around 2009, that the debate on GDP-Is has gained momentum as witnessed by the rapidly increasing number of papers discussing the topic.

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1 The author wishes to thank Mark Joy and colleagues from the Bank of England for a thoughtful discussion of a preliminary version of this paper as well as participants to an internal seminar in Bank of Italy. The authorization by Bank of England to re-print Chart 1 is gratefully acknowledged. The views expressed in this paper are solely those of the author and do not necessarily reflect those of the Bank of Italy or of the Eurosystem.

2 A (non-exhaustive) survey of the literature could include Borensztein and Mauro (2004), Schröder, Heinemann, Kruse, Meitner (2004), Chamon and Mauro (2006), Griffith-Jones and Sharma (2006), Ruban, Poon and Vonatsos (2008), Sandleris, Sapriza and Taddei (2008), Durdu (2009), Ghosh, Kim, Mendoza, Ostry and Qureshi (2011), Missale and Bacchiocchi (2012), Brooke, Mendes, Pienkowski and Santor (2013), Miller and Zhang (2013), Barr, ...

However, despite such efforts, there is still very limited issuance by sovereigns of financial products linked to GDP worldwide (Argentina is one exception). The reason why deeds have not followed words is not obvious. On the one hand, the rationale for such bonds, as sketched above, seems straightforward. On the other hand, some oft-mentioned criticisms to GDP-Is – e.g. the complexity of pricing, the first mover argument and the risk of GDP data manipulation by the issuer who happens to be also in command of the national statistical agency – can in turn be challenged, at least in qualitative terms. Nor does it help to reconcile the gap between theory and practice the fact that the literature is not rich in dissenting voices on GDP-Is.

Against this background, the motivation of this paper is to put forward an internally-coherent analytical framework to discuss what GDP-Is could bring to public debt management, letting the cold language of math to frame the issues (this is the “formal” element of the research mentioned in the title). As ubiquitous in public debt management, the analysis focuses on the trade-off between costs and risks: how much the cost of serving public debt could rise as a result of the supply of GDP-Is and to what extent these bonds could reduce the risk borne by the Treasury that shocks would inflate the debt-to-GDP ratio to the point of triggering a crisis of confidence in the sustainability of public finances.

In doing so, the paper introduces two elements of novelty compared to the extant related literature (as far as we are aware of). Firstly, the paper solves the portfolio allocation problem assuming that the argument of the utility function maximized by the representative agent includes not only the return from this portfolio but also, and this is the novelty, a proxy of his non-financial income. The objective here is to address what is probably the main challenge ahead of a successful launch of GDP-Is over a significant scale. In effect, especially the domestic investor would take on a double risk if he invests heavily in GDP-Is issued by his own Treasury: when GDP grows less than expected, the odds are that he would observe a decline in his non-financial income coupled with a lowering financial return on the GDP-Is themselves. Hence, it seems important to study how the co-

3 As regards the pricing, traders would need to factor in the uncertainty over the outturn in GDP. However, financial markets have proved to be apt at quoting instruments far more complex that those linked to developments in national income. Next, there is a first mover argument. No sovereign of adequate standing and decent size seems eager to issue GDP-Is before a liquid market is established. Clearly, this circular argument annihilates the chances of creating a liquid market in the first place. It is also an odd point to raise however since if there is something financial markets are good at is innovation. It is not certain either that the hype surrounding the risk of data manipulation by the issuer – the argument being that the sovereign could fix GDP figures to pay less on the service of GDP-Is – is fully justified. On the one hand, the risk of data manipulation exists anyway whatever the type of bonds issued to finance public debt and this risk has not stood in the way of the increasing supply of so-called linkers, i.e. bonds indexed to inflation. On the other hand, at least in a democratic country where the standing government is going to be tested in general elections sooner or later, it does not seem obvious that such a government wishes to describe national economic conditions worse than reality.

4 One partial exception is Zhang (2011), whose paper however is not completed.
movement between financial and non-financial income affects the demand for government securities.\(^5\)

Secondly, the paper examines which impact, if any, the supply of GDP-Is may have on the condition of issuance by the Treasury on its plain vanilla bonds. The intuition runs as follows. If GDP-Is actually deliver what proponents see in them – an insurance-type contract which loosens the impact of unforeseen recessions on the sustainability of public finances – then it seems reasonable to assume that investors would request a lower credit risk premium across the board, on both nominal and GDP-I bonds. In fact, as we will find out through the research, there may be also a flip side of the coin since the offer of GDP-Is could lower the demand for nominal bonds, bringing about an increase in their yields. Analytically, these impacts are examined by widening the choice of the representative agent in his portfolio allocation to a third (risk free) asset issued by some third party, besides the two lines of securities issued by the Treasury. Namely, the model does not constrain the overall demand for the two government securities to be fixed – de facto, an implicit assumption in models where the choice is only between GDP-Is and plain vanilla bonds – but rather the model let it to fluctuate.

It helps to state the don’ts of this paper, besides the dos. To start with, this paper focuses mostly on analytical aspects of GDP-Is while we don’t add much in terms of general, qualitative introduction on these securities, besides what said in this section. Much and well has already been written among others by Barr, Bush and Pienkowski (2014), Brooke, Mendes, Pienkowski and Santor (2013) and Missale and Bacchiocchi (2012). Nor does this paper dwell on details of the indexation formulae, fearing that we wouldn’t see the forest for the trees. Last but certainly not least, the paper discusses at length GDP-Is once these bonds have become an established feature, while it skips the issues the Treasury would need to manage in the transition phase. This is not at all to neglect how important these issues are – if only one thinks to the hurdle of ensuring a liquid market for the GDP-Is in the initial phase – but it is rather to avoid over-ambition on what a single paper can achieve. Clearly, the design of the starting phase sets out the agenda for future research.

The rest of the paper is organized as follows. Section 2 lays down the set-up of the model. There is nothing extremely new but this section defines the main symbols and concepts. Section 3 offers some preliminary results on the fiscal reactivity, the maximum level of sustainable debt, the probability of default and the clearing level of the interest rates. Section 4 introduces the GDP-Is and includes some brief remarks on indexation formulae. Sections 5 solves the portfolio allocation problem (in fact, problems since we will deal with a “narrow” and a “broad” problem). Section 6 deals with the capability of GDP-Is to loosen the link going from debt and growth shocks to the probability of default Section 7 concludes.

\(^5\) This double risk could be less critical for a foreign investor provided the GDP in his resident country is not highly correlated with that of the country issuing the bond. In fact, if this correlation were negative, then for a foreign investor the purchase of GDP-Is would represent a form of hedging to his non-financial income.
Chart 1

**Real rates of change of euro area GDP: outturn and forecasts**

*per cent*

Source: Eurostat and IMF

Chart 2

**Gross government debt under either conventional or GDP-linked bonds (100 basis points premium): for an indebted advanced economy (1)**


(1) the chart shows debt to GDP ratio paths corresponding to 1st, 50th and 99th percentiles of the joint normal distribution of shocks. The orange line shows the 50th percentile path for conventional debt. The green line shows the 50th percentile path for GDP-linked debt.
2. The set-up of the model

We follow here, with some adaptations, Gosh, Kim, Mendoza, Ostry and Qureshi (2011) and Barr, Bush and Pienkowski (2014). The set-up of the model is organized around four equations describing respectively the debt-to-GDP ratio, the interest rate on a government security as a function of the return of a risky asset, the rate of change of the GDP and, finally, the primary surplus.

The law of motion of the debt-to-GDP ratio.

\[
d_t - d_{t-1} = \frac{r_t - g_t^d}{1 + g_t^d} d_{t-1} - s_t + v_t \quad \text{E}(v_t) = \sigma^2_v \quad \text{Var}(v_t) = \sigma^2_v
\]

\[
d_t = \frac{1 + r_t}{1 + g_t^d} d_{t-1} - s_t + v_t
\]

where \(d_t\) is the debt-to-GDP ratio, \(r_t\) the average cost per unit of debt, \(g_t^d\) the rate of change in nominal GDP of the debt issuing country, \(s_t\) the surplus-to-GDP ratio, and \(v_t\) a shock of debt as ratio to GDP. While equations (1a)/(1b) are well established, a few remarks are nevertheless in order. Firstly, the interest rate \(r_t\) standing for the cost of debt is not the same concept as the interest rate at which new government securities are issued. Secondly, the concepts of interest rates and GDP change which drive result (1a) could be defined alternatively in “real” terms (i.e. changes in GDP net of dynamics in underlying prices) or in nominal, headline ones. In this paper we opt for the latter option as the nominal concept is more customary in portfolio allocation problems and this is a core item in the current research. Thirdly, at this level of generality we do not need to specify in detail the stochastic process followed by the shock \(v_t\), besides defining its first and second moments.

The baseline equation of the interest rate on the risky asset

\[
r_t = r_t^F + \frac{p_t}{1 - p_t} (1 + r_t^F) (1 - \theta) \equiv r_t^F + P_t
\]

\footnote{The details of the derivation of results are in Annex. Even there, to save space, we omit steps which involve only sheer calculus and no substitutions, new assumptions and the like. Full proofs are available upon request.}

\footnote{This is the public budget net of expenditures for interest on outstanding public debt.}

\footnote{Accordingly, it would be more appropriate to use a symbol other than ‘\(r\)’ to refer to interest rates set in auctions, to single out what is the cost borne out by the Treasury on new securities from the cost on the outstanding stock. However, one down side of the richness of results which may be gathered from the framework put forward in this paper is rather lengthy and not always so intuitive algebra, at times. Hence the need for simplifying assumptions whenever these should not cause loss of generality and meaning of symbols can be understood from the context.}

\footnote{In addition, we reckon there is limited loss in the general understanding of the problem of GDP-Is in public debt management so long as the choice is between GDP-Is and nominal bonds. Of course, should a broader model be developed – in which the choice is between GDP-Is and nominal bonds and inflation-linked bonds – then it would be appropriate to think in terms of real concepts and to model inflation.}
where $r_t$ is as above, $r^F_t$ is the return on a risk-free asset (in general, sovereign securities of advanced economies are no longer necessarily perceived as such) and the second addendum $P_t$ is the credit risk premium as a function of the probability of default $p_t$ and the recovery rate $\theta$ in case of default.\textsuperscript{10}

\[ GDP \text{ dynamics} \]

\[ g^I_t = E(g^I_t) + \epsilon_t \quad E(\epsilon_t) = 0 \quad \text{Var}(\epsilon_t) = \sigma^2_\epsilon \quad (3a) \]

where $E$ is the expectation operator and, in effect, (3a) simply states that GDP forecasts are subject to errors. Just to fix the ideas, it helps to assume that forecasts are based on a simple AR(1) process so that the conditional and unconditional expectations of the GDP rate of change are respectively\textsuperscript{11}

\[ E(g^I_t) = \delta + \phi g^I_{t-1} \quad |\phi| < 1 \quad (3b) \]

\[ E(g^I_t) = \frac{\delta}{1-\phi} \quad (3c) \]

\[ The \ primary \ surplus \]

\[ s_t = \min\{\beta \left[ (r_t - g^I_t)/(1 + g^I_t)\right] d_{t-1}, \gamma\} \quad \beta \geq 0 \quad \gamma > 0 \quad (4) \]

This expression posits a reaction function by the policy maker which sets the primary surplus as of time $t$ depending on the debt ratio as of time $t-1$ as well as the spread $(r_t - g^I_t)$ between the cost of debt and GDP change, filtered through a parameter $\beta$. The level of the latter conveys how strictly the policy maker wishes to stabilize the debt ratio: if $\beta < 1$ he accepts the debt ratio $d_t$ to be inversely related to the primary surplus $g^I_t$, if $\beta > 1$ a direct relationship is established between the ratio and the primary surplus while if $\beta = 1$ the policymaker steers the primary surplus to offset fully the impact of $r_t$ and $g^I_t$ so that the debt ratio fluctuates only because of the shock $\nu_t$ (details are in Annex 1).\textsuperscript{12} Quite crucially, result (4) foresees that the primary surplus cannot be larger than a level $\gamma$, which can be understood as a threshold of “fiscal fatigue” meaning the maximum level of austerity a policy maker can impose to his constituency, at least in a democratic country (Ghosh, Kim, Mendoza, Ostry and Qureshi, 2011, and Miller and Zhang, 2013).

\textsuperscript{10} As well known, result (2) is derived under an arbitrage condition according to which the return on the risk free asset is equal to the one on the risky asset under the alternatives, weighed according to $p_t$ and $(1- p_t)$, that the credit risk materializes or not:

\[ 1 + r^F_t = (1 - p_t)(1 + r_t) + p_t \theta (1 + r^F_t) \]

\textsuperscript{11} It goes without saying that one can think to more complex structures than an AR(1), which is selected purely for illustrative purposes. Again, the compelling need to keep simple whatever is not strictly necessary to understand the problem of GDP-Is informed this choice too.

\textsuperscript{12} One could also think to a first variant of (4) such that the debt ratio increases whenever the rate of change in GDP falls below a threshold which is not necessarily equal to zero. In a second variant of (4) the reaction function is designed so as to cater for the full absorption of any shock and, as a result, the debt ratio is constant over time.
3. Some selected results derived from the four basic equations

3.1 The maximum level of sustainable debt

By imposing $d_t = d_{t-1}$ in (1b) and solving for $s_t$, one derives the primary surplus that keeps the debt ratio stable:

$$s_t^e = \frac{r_t - g_t^I}{1 + g_t^I} d_{t-1} + v_t$$

(5)

where the “$e$” signals that we are referring to an equilibrium level. In turn, the highest level $\bar{d}$ at which the debt ratio can be kept stable by maneuvering the primary surplus is derived by imposing the condition $s_t^e = \gamma$ in (5) and solving for the debt ratio

$$\bar{d} = \frac{1 + g_t^I}{r_t - g_t^I} (\gamma - v_t)$$

(6)

This simple expression is remarkable in many ways. To mention one, we tend to think of the highest sustainable level of the debt ratio as some constant at least approximately; in fact, it is time variant (although to shorten notation we do not explicitly write the time pedix) being a function of $g_t$, and $r_t$ besides the debt shock $v_t$. Moreover, the maximum debt level $\bar{d}$ is also a function of the threshold of fiscal fatigue $\gamma$. As the latter can hardly be measured with much precision, equation (6) speaks volumes on why it is difficult to gauge reliable figures for the maximum sustainable debt in the first place. By the same token, when the cost of debt is close to the change in nominal GDP so that the denominator $(r_t - g_t^I)$ is close to zero, the right hand-side of (6) can fluctuate markedly even for small changes in the other parameters. Finally, $\bar{d}$ is inversely related to the cost of debt $r_t$, which suggests that the central bank may buy “space” in terms of achieving higher levels of $\bar{d}$ itself by pushing $r_t$ downwards, e.g. via extensive purchase of government securities. This could offer an intuition on the apparent conundrum of the sustainability of the Japanese public debt, in spite of its extreme high levels and only moderate change in nominal GDP.

3.2 The probability of default

We define the probability of default as the probability that in time $t$ the debt-to-GDP ratio is higher than the threshold $\bar{d}$, conditional that it was below at in time $t-1$:

$$p_t = \text{prob}(d_t > \bar{d}|d_{t-1} < \bar{d})$$

(7)

Note that reaching a level of the debt ratio higher than the said threshold is not technically a state of default yet. In fact, $d_t > \bar{d}$ describes a state of the world where the policy maker avails of no further room to increase the fiscal surplus with the goal of reining in public finances. However, lucky

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13 The time variability of $\bar{d}$ holds even if one understands $r_t$ and $g_t^I$ as averages through the business cycle, rather than values referred to a specific point in time, since i.a. even these averages fluctuate from one cycle to the next.
combinations of debt and/or GDP shocks may still come to the rescue, by lowering the debt ratio without further increases in the surplus $s_t$.

Through substitution of (1b) and (6) in (7), one obtains

$$p_t = \text{prob} \left[ d_{t-1} - s_t + v_t > \frac{1+g_t}{r_t-\delta_t} (\gamma - v_t) \bigg| \bar{d} \right]$$

(8a)

After some algebra and accepting an approximation up to a term in the second power of the GDP shock $\varepsilon_t$ (the approximation should be negligible for most values of the shock; details are in Annex 2), the probability of default can be written as

$$p_t = \text{prob} \left\{ v_t - \frac{1-r_t+2E(g_t)}{[1+E(g_t)]^2} d_{t-1} \varepsilon_t > \gamma - \frac{r_t-E(g_t)}{1+E(g_t)} d_{t-1} \bigg| \bar{d} \right\}$$

(8b)

were the term before the inequality sign ‘$>$’ is a function of debt and GDP shocks, while the term afterwards is a combination of the fiscal fatigue parameter $\gamma$, the (unit) cost of the outstanding stock of debt, the debt ratio and the parameters driving the change in GDP. Result (8b) will prove handy when studying the role GDP-I can play in loosening the link from two types of shocks to changes in the probability of default.\textsuperscript{14}

Note also that neither (8a) nor (8b) represent final solutions, insofar the interest rate $r_t$ is itself a function of the probability $p_t$. If for whatever reason the belief that $p_t$ ought to rise gains ground among market participants, and this higher level is priced in the interest rate asked to underwrite the bonds issued by the Treasury, then it is straightforward to derive from (8b) that an increase in $r_t$ translates into a higher level in terms of $p_t$. That is the new market belief, be that based on sound fundamentals or animal spirits, is self-fulfilling. A well-managed public debt management unit handles the impact of such rollercoaster in market’s mood by keeping long enough the average duration of the outstanding stock of government securities. By doing so, should banks suddenly seek higher interest rates in new auctions, this would have only a limited impact on the overall cost of debt. Conversely, a Treasury which manages a public debt with short maturity is critically exposed to changes in market sentiment.

\textbf{4. Cash flows and period payments in GDP-I bonds}

\textit{4.1 The cash flows of the two bonds}

We now turn to the mix of nominal and GDP-I bonds issued by the Treasury to refinance public debt. For the purpose of this paper, we understand as ‘nominal’ or ‘plain vanilla’ a bond which pays at maturity to its holder an interest rate already set at the original auction; conversely, a GDP-I bonds

\textsuperscript{14} Neither (8a) nor (8b) are in effect final solutions, insofar the interest rate $r_t$ is itself a function of $p_t$ according to (2). If for whatever reason market participants shares the belief that $p_t$ is to increase, then within (8b) the ratio before $(d_{t-1}, \varepsilon_t)$ increases as well while the one before $d_{t-1}$ after the inequality sign “$>$” decreases. That is, the belief that $p_t$ is to increase yields an increase in the result of (8b), i.e. $p_t$ in a self-fulfilling process. In the real world a Treasury manages such “animal spirits” from the market by lengthening the maturity of debt. In this way, the higher $p_t$ applies only to new gross issues and feeds only gradually in the cost $r$, of the whole outstanding stock of debt.
bondholder does not know in advance the rate of interest he will cash in at maturity. In a baseline scenario, in both lines the Treasury pays only one coupon at maturity and the normalized structure of cash flow looks as follows

\[
\begin{array}{c|c|c}
 t_0 & t_1 \\
\hline
\text{nominal bond} & -1 & 1 + r_t^N \\
\text{GDP-I (only coupon is indexed)} & -1 & 1 + r_t^{G,p} \\
\text{(both coupon and principal)} & -1 & (1 + g_{1}^{I}) + r_t^{G,p}
\end{array}
\]

where \( r_t^N \) denotes the interest rate clearing the auction of the nominal bond and paid ex post, and \( r_t^{G,p} \) the interest rate paid ex post by the Treasury on the GDP-I bond.

Without much loss of generality, one can think to the two securities as zero-coupon bonds whose auctions are held in close sequence. First, banks submit bids for the nominal bond in terms of the interest rate; of note, individual banks’ bids reflect heterogeneous beliefs on the probability of default as we are in a world where the interest rate is defined by result (2). Once the result \( r_t^N \) of the auction is determined, banks which have submitted successful bids are expected to pay \((100 - r_t^N)\) to underwrite the bond, with adaptations in this formula should the bond be longer than one year.

Shortly afterwards, the Treasury holds the auction of the GDP-I bond. The way banks bid at this second auction depends on two drivers. Due to arbitrage, investors expect the same return for the GDP-I and nominal bonds, so that under risk neutrality and if the indexation applies only to the coupon, the following relation holds:

\[
1 + E(r_t^{G,p}) = 1 + r_t^N \tag{9a}
\]

The second driver is the commitment by the Treasury to pay at maturity on this bond the interest rate which was set at the auction plus a spread between the realized rate of change in change and a pre-announced level \( g \)

\[
r_t^{G,p} = r_t^{G,a} + g_{1}^{I} - \bar{g} \tag{9b}
\]

Taking expectations on both sides of (9b)

\[
E(r_t^{G,p}) = r_t^{G,a} + E(g_{1}^{I}) - \bar{g} \tag{9c}
\]

\[
\Rightarrow r_t^{G,a} = r_t^N - [E(g_{1}^{I}) - \bar{g}] \tag{9d}
\]

Result (9d) describes how banks will participate at the GDP-I auction, with differences across bids depending on participants’ views on the expected rate of change in the GDP. Indeed, both the interest rate \( r_t^N \) of the nominal bond and the fixed level \( \bar{g} \) are public information at the time this second auction is being held. Banks whose bid is successful will pay \((100 - r_t^{G,a})\) to underwrite the bond while ex post the Treasury will pay
\[ r_t^{G:p} = r_t^{G:a} + g_t - \bar{g} = r_t^N + \varepsilon_t \]  

(9e)

A few words are worth being said on \( \bar{g} \). Result (9e) shows that the specific choice of this parameter does not affect the cost borne out by the Treasury (at least in a risk neutral environment). It is true that this parameter does enter result (9d) for the interest rate \( r_t^{G:a} \), but then all banks need to do to adapt mechanistically their bids in accordance with the Treasury choice of \( \bar{g} \). From a different perspective, the specific choice of \( \bar{g} \) may prove to be relevant, bearing in mind that in the settlement of the security, successful banks will pay \( (1 - r_t^{G:a}) \) or, which is the same, \( 1 - r_t^N + [E(g_t) - \bar{g}] \). That is by setting a “high” \( \bar{g} \), the Treasury defines the conditions to receive a “low” inflow of cash, all other things being equal.\(^{15}\) Alternatively the Treasury could set \( \bar{g} \) around the consensus forecast for the change in GDP, so that the price of the bond at the auction will be approximately \( (1 - r_t^N) \). The snag is that should the realization of the shock \( \varepsilon_t \) be negative and fairly large in absolute value while, at the same time, the interest rate \( r_t^N \) close enough to nil, then what the Treasury will pay at maturity, i.e. \( (1 + r_t^N + \varepsilon_t) \), could turn out to be below unity. Certainly, in doing so the Treasury is simply abiding by the GDP-I contract. However the payment at maturity of a sum below par for principal and coupon combined sounds ominously close to an haircut in a default. A prudent Treasury may thus wish to prevent such perception to get even near to materialize and it could do so by setting \( \bar{g} \) clearly below \( E(g_t) \). At this stage this is all highly speculative but it explains the choice to retain explicitly the symbol \( \bar{g} \) throughout our equations – also for future expansions of the research – although as noted above in a risk neutral environment there would be no loss of generality in imposing \( \bar{g} = 0 \).

\[ (1) \]

\[ (2) \]

4.2 The arbitrage condition under risk neutrality

The exact relationship between the interest rates \( r_t^N, r_t^{G:p} \) and \( r_t^{G:a} \) changes depending on a number of factors: (i) agents are risk neutral or risk averse; (ii) indexation applies only to the coupon or both coupon and principal; (iii) the issuer pays the coupon only at maturity or it pays several intra-period coupons; (iv) arbitrage condition holds between the market for the securities or segmentation applies. In this section we deal briefly with topics (ii) and (iii), while we will discuss extensively topic (i) in the remainder of the paper. Conversely, we do not really examine topic (iv) on the premise that we are studying GDP-Is in the steady state and the odds are that this state is reached if the markets for plain vanilla and GDP-Is have got actually integrated (besides the general aim to keep the paper focused).

If the indexation of the GDP-Is applies both to the coupon and the principal, the following arbitrage condition holds between the interest rates set in the auctions of the GDP-I and nominal bonds:

\[ r_t^{G:a} = r_t^N - 2 [E(g_t) - \bar{g}] \]  

(10a)

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\(^{15}\) The reduction in the amount settled may be quite large: if the (zero-coupon) GDP-I has a maturity of five years each increase in \( \bar{g} \) by 20 basis points translates into a lower price received by the Treasury by one full percentage point. If the maturity of the bond is 10 years, the reduction in the price reaches two percentage points.
that is, results (10a) and (9d) differ only by a factor of 2. If two periods of payment are involved and indexation applies again only to the coupons, the following holds

\[ r^G_t = r^N_t - \frac{(1+p)|E(g_{t1}) - \bar{g}| + |E(g_{t2}) - \bar{g}|}{(2+\rho)} \]  

(10b)

where \(E(g_{t1}^I)\) and \(E(g_{t2}^I)\) are the expected change in GDP, expectation taken at the time of issuance of the bond, referring respectively to the period of the first and second coupon respectively while \(\rho\) is a factor of discount. Again (10b) resembles (9d) except that the expression relating to the expectation gets somewhat more complex. Note incidentally that it is fairly straightforward to generalize (10b) for \(N>2\) periods (though at the cost of longer algebra). Finally if indexation applies to both coupon and principal, one has

\[ r^G_{t0} = r^N_t - \frac{(2+p)|E(g_{t1}^I) - \bar{g}| + 2E_{t0}[E(g_{t2}^I) - \bar{g}]}{(2+\rho)} \]  

(10c)

and once more the structure of (10c) resembles that of (9d) (details are in Annex 3). As a note of caution, these results apply to risk neutrality and the algebra is more complex when risk aversion holds.

5. Portfolio allocation under risk aversion

As hinted at in the introduction, we organize the portfolio allocation problem in two steps: first we discuss a “narrow” portfolio where the choice is between the two lines of securities (GDP-I and nominal bonds) issued by the Treasury as manager of the public debt; then, we turn to a “broad” portfolio which includes beside the “narrow” portfolio also a third risk-free asset, not issued by the same Treasury.

5.1 The “narrow” portfolio problem

In this problem the representative agent invests in GDP-I and nominal bonds according to weights \(\alpha^G \in [0,1]\) and \(1-\alpha^G\) respectively. Note that this is not a fully standard portfolio allocation problem in two respects. Firstly, the choice is between two assets which are both risky: the nominal bond carries a credit risk while the GDP-I carries both credit risk\(^{16}\) and risk deriving from the uncertainty on the outturn on GDP.\(^{17}\) Secondly, the unknown of the problem is not the weight \(\alpha^G\), which is set by the Treasury, but rather the premium the representative investor will seek on the GDP-I per each unit of additional risk taken on compared to the nominal bond.

The investor’s choices will be set by the maximization of a CRRA utility function, which as well known is described as

---

\(^{16}\) To keep the problem tractable, we assume here that a pari passu clause applies. Namely, in the event of default the same hair-cut will be applied on the two lines of securities.

\(^{17}\) For the sake of simplicity we assume that the debt manager is committed to a pari passu clause, i.e. it will handle the two securities equally in case of default, and this commitment is perceived as credible by the investors.
\[
U(c) = \begin{cases} 
\frac{1}{1-\tau}c^{1-\tau} & c > 0, \ c \neq 1 \\
\ln c & c = 1 
\end{cases}
\] (11)

and which we will develop through a standard Taylor expansion up to the second order (Campbell and Viceira, 2001).\(^{18}\) Next and as a central point of this research, the agent draws the utility \(c\) both from its investment in securities and from more general sources of income:

\[
c \equiv r_t - r^F_t + E(g^B_t)
\] (12)

where \(g^B_t\) denotes the change in the non-interest income of the representative bondholder, which is in turn the weighted average of the rates of change of the GDPs of the home country and the ‘rest-of-the-world’ in proportion to \(\omega\) and \((1-\omega)\)\(^{19}\)

\[
g^B_t = \omega g^W_t + (1-\omega)g^I_t \quad \omega \in [0, 1]
\] (13)

The return \(r_t\) on the portfolio of risky assets is the weighted average according to weights \(\alpha^G\) and \((1-\alpha^G)\) respectively of the return on the GDP-Is and that on the nominal bonds

\[
r_t = \alpha^G r^{G,a}_t + (1-\alpha^G)r^N_t
\] (14)

In turn, in a risk aversion environment we write down \(r^{G,a}_t\) adapting (9d):

\[
r^{G,a}_t = r^N_t - [E(g^I_t) - \bar{g}] + \eta \tau \sigma^2_e
\] (15)

where the third term on the right hand side is the premium on account on the uncertainty in the outturn of the GDP, broken down in the product of the ‘quantity of risk’ \(\sigma^2_e\) (the variance in the shock \(\varepsilon\)), degree \(\tau\) of risk aversion and the monetary value \(\eta\) of unit of risk.

For the more general case of \(c > 0\) and \(c \neq 1\), up to an approximation which can be proved to be of limited scale and after some lengthy algebra (details are in Annex 4), the solution for the optimal value \(\hat{\eta}\) is derived as

\[
\hat{\eta} = \frac{\tau^{1/2}(1 + \tau)^{1/2}[P_t - \alpha^G[E(g^I_t) - \bar{g}]] + 2E(g^B_t)}{[\tau^{1/2}(1 + \tau)^{1/2} - 2]\alpha^G \tau \sigma^2_e}
\] (16a)

or by substitution of (13)

\[
\hat{\eta} = \frac{\tau^{1/2}(1 + \tau)^{1/2}[P_t - \alpha^G[E(g^I_t) - \bar{g}]] + 2\omega E(g^W_t) + 2(1-\omega)E(g^I_t)}{[\tau^{1/2}(1 + \tau)^{1/2} - 2]\alpha^G \tau \sigma^2_e}
\] (16b)

---

\(^{18}\) In principle, there could be merits in considering also higher moments, say up to the fourth, but the algebra becomes too lengthy and cumbersome.

\(^{19}\) In a more precise but also longer fashion, the weight \(\omega\) applies to the weighted average of the rate of change in GDP of the countries of residence of the foreign investors, in proportion to their holdings. As a further remark, we assume for the sake of simplicity that same pairs of weights \([\omega; 1-\omega]\) apply to both nominal and GDP-I bonds.
Remarkably, the monetary premium \( \hat{h} \) is generally but not necessarily always positive. Note first that the denominator of the ratio in (16a)/(16b) is positive for \( \tau > (\sqrt{17} - 1)/2 \). This looks a very mild condition following Janeczek (2004), who reckons that the average investor’s risk aversion parameter in a CRRA is in order of 30 while few investors with enough experience could exhibit aversion below 20. Provided this condition on the parameter of risk aversion is met, then

\[
\hat{h} > 0 \iff \tau^{1/2}(1 + \tau)^{1/2} \{ P_t - \alpha^G [E(g_{it}^1) - \bar{g}] \} + 2\omega E(g_{it}^W) + 2(1 - \omega) E(g_{it}^T) > 0 \quad (17)
\]

In turn, the second condition is verified for

\[
0 < \alpha^G < \frac{P_t}{E(g_{it}^1) - \bar{g}} + \frac{2\omega E(g_{it}^W) + 2(1 - \omega) E(g_{it}^T)}{\tau^{1/2}(1 + \tau)^{1/2}[E(g_{it}^1) - \bar{g}]} < 1 \quad (18)
\]

Note that the if the sum

\[
\frac{P_t}{E(g_{it}^1) - \bar{g}} + \frac{2\omega E(g_{it}^W) + 2(1 - \omega) E(g_{it}^T)}{\tau^{1/2}(1 + \tau)^{1/2}[E(g_{it}^1) - \bar{g}]}
\]

is either negative or greater than +1, then any value of \( \alpha^G \in [0,1] \) fulfils the conditions laid down in (18), that is \( \hat{h} \) is always strictly positive. For instance, this happens whenever the credit risk premium \( P_t \) exceeds the expected change in nominal GDP \( E(g_{it}^1) \) of the issuing country. More generally, given the expected domestic and “rest-of-the-world” GDP rates of change, the higher is \( P_t \), the narrower (and closer to unit) is the range of values \( \alpha^G \) must take to yield a negative \( \hat{h} \). Clearly, the intuition is that countries with low or mediocre creditworthiness will have to accept that should they issue GDP-Is, then investors will charge a positive \( \hat{h} \) to underwrite the bonds. Numerical examples will be offered later in this section.

It is of interest to discuss some partial derivatives of \( \hat{h} \):

\[
\frac{\partial \hat{h}}{\partial \alpha^G} = -\frac{[\tau^{1/2}(1 + \tau)^{1/2}P_t + 2\omega E(g_{it}^W) + 2(1 - \omega) E(g_{it}^T)]^2}{[\tau^{1/2}(1 + \tau)^{1/2} - 2] \tau \sigma_\epsilon^2 (\alpha^G)^2} \quad (19)
\]

This partial derivative is negative under the mild condition \( \tau > (\sqrt{17} - 1)/2 \), already discussed above. Namely the optimal premium \( \hat{h} \) is positive (usually but not necessarily always) and decreasing the higher is the weight \( \alpha^G \). From this perspective if a Treasury wishes to launch the GDP-Is, it should rather do so on a large scale although result (19) also highlights that the marginal reduction in \( \hat{h} \) decreases as \( \alpha^G \) gets large and in any case is an inverse function of the risk aversion parameter \( \tau \). That is the reduction in \( \hat{h} \) achieved through a large supply of GDP-Is may turn out more marginal in a risk off environment.

Turning to the partial derivative of \( \hat{h} \) w.r.t. \( P_t \),

\[
\frac{\partial \hat{h}}{\partial P_t} = \frac{\tau^{1/2}(1 + \tau)^{1/2}}{[\tau^{1/2}(1 + \tau)^{1/2} - 2] \alpha^G \tau \sigma_\epsilon^2} (1 - \theta)(1 + \tau^F) \frac{1}{(1 - \rho_\epsilon)^2} > 0 \quad (20)
\]
Hence, the condition $\partial \hat{\eta} / \partial p_t > 0$ holds under the same mild conditions we identified above. That is, the premium for the GDP-related risk increases with the probability of default unless there is a fairly extreme risk on environment.

Third and finally, the partial derivatives of $\hat{\eta}$ w.r.t. to the change in GDP is

$$\frac{\partial \hat{\eta}}{\partial E(g_t^I)} \Bigg|_{\partial E(g_t^W)/\partial E(g_t^I) = 0} = \frac{-\tau^{1/2}(1 + \tau)^{1/2}\alpha_G + 2(1 - \omega)}{\tau^{1/2}(1 + \tau)^{1/2} - 2\alpha_G\tau\sigma_\varepsilon^2}$$  (21a)

The algebra of (21a) says the this partial derivative may take either sign, where from the Treasury’s viewpoint it would be more convenient if the positive sign could prevail. The intuition is that it is acceptable to pay to bondholders a higher $\hat{\eta}$ when the rate of change in national income increases and viceversa. Given $\tau > (\sqrt{17} - 1)/2$, (21a) is positive if

$$\alpha_G > \frac{2(1 - \omega)}{\tau^{1/2}(1 + \tau)^{1/2}}$$  (21b)

Say, if $\tau = 30$ and $\omega = 0.5$, then condition (21b) translates into $\alpha_G > 3.28\%$ while if $\omega = 0.3$ the condition is $\alpha_G > 4.59\%$. Namely, the Treasury needs to cap the issuance of GDP-Is if it wishes to established a positive relation between the premium $\hat{\eta}$ and the expected change $E(g_t^I)$ in national income.20 As we will see later, this goal sets however a trade-off in terms of the other goal of having GDP-Is which bring about a material mitigation in the debt ratio stabilization faced with shocks, since the latter goal calls for a much larger weight $\alpha_G$, say at least 30% if not higher.

From the point of view of the investors, a possible economic interpretation of the negative sign of the partial derivative laid down in (21a) when $\alpha_G$ is only relatively large would run as follows. If investors can hedge their utility through an increasing non-financial income, they will stand ready to take on more risk on financial investments. Or, which is the same, they will underwrite the GDP-Is accepting a lower remuneration per unit of GDP-related risk. Conversely, the lower is $E(g_t^I)$, the higher should be $\hat{\eta}$ given that especially the domestic investors will be very keen to be well remunerated against the “double risk” (on financial and non-financial income) they are taking on at a time of declining economic fortunes.

A further observation about (21a) is that this result is derived assuming that the rate of change of the “rest-of-the-world” GDP does not change with the one of the domestic GDP. In fact, among large advanced economies, growth tends to be positively correlated, with a coefficient of correlation in the order of 0.7 if measured on real rates of change and 0.5 if measured on nominal rates of change (Tables 1 and 2 overleaf). Hence, we work out a variant of (21a) where we relax this constraint assuming more generally that $\partial E(g_t^W)/\partial E(g_t^I) \equiv \rho > 0$. The new partial derivative of $\hat{\eta}$ w.r.t. to $E(g_t^I)$ is

$$\frac{\partial \hat{\eta}}{\partial E(g_t^I)} \Bigg|_{\partial E(g_t^W)/\partial E(g_t^I) = \rho} = \frac{-\tau^{1/2}(1 + \tau)^{1/2}\alpha_G + 2(\omega \rho + 1 - \omega)}{\tau^{1/2}(1 + \tau)^{1/2} - 2\alpha_G\tau\sigma_\varepsilon^2}$$  (21c)

The equivalent of condition (21b) is

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20 To gauge how large are the above thresholds in terms of $\alpha_G$, it may help to bear in mind that e.g. the BTP indexed to euro area inflation accounted for 11.6% of the stock of Italy’s government securities, where the Italian Treasury ranks among the most active issuers of these linkers.
\[ \alpha^G > \frac{2(\omega \rho + 1 - \omega)}{\tau^{1/2}(1 + \tau)^{1/2}} \]  

(21d)

Using again \( \tau = 30 \) and \( \omega = 0.5 \), then the above numerical result of 3.28% rises to 4.92% (if \( \omega = 0.3 \) then the result of 4.59% rises to 5.57%). Namely a positive correlation in domestic and foreign GDPs expands the domain of weights \( \alpha^G \) of GDPs within which a positive correlation between \( \eta \) and \( (\bar{g^1}) \) prevails. An intuition on this result wants that if there is such correlation, then GDP-Is lose some of their appeal for foreign investors as an hedging instrument. In turn, this means that GDP-Is will tend to be more costly (compared to a world where GDP-Is are uncorrelated) since the foreign investors too are subject to the aforementioned “double risk”, like domestic ones even if on a smaller scale so long as \( \rho < 1 \), and seek an adequate remuneration.

### Table 1

**GDP: correlation between real rates of change**  
*(based on yearly data, 1992-2016)*

<table>
<thead>
<tr>
<th></th>
<th>France</th>
<th>Germany</th>
<th>Italy</th>
<th>Japan</th>
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<th>USA</th>
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<tr>
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<td>0.74</td>
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Memorandum item: average of pairwise coefficients (with Japan) = 0.69; average of pairwise coefficients (without Japan) = 0.75.

Source: elaboration on OECD data

### Table 2

**GDP: correlation between nominal rates of change**  
*(based on yearly data, 1992-2016)*

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<th></th>
<th>France</th>
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<th>Japan</th>
<th>UK</th>
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<td>USA</td>
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<td>0.14</td>
<td>0.41</td>
<td>1</td>
</tr>
</tbody>
</table>

Memorandum item: average of pairwise coefficients (with Japan) = 0.48; average of pairwise coefficients (without Japan) = 0.64.

Source: elaboration on OECD data

We can now derive the results in terms of the cost borne out by the Treasury owing to the issuance of GDP-Is. After some algebra one obtains

\[ r_t^p - r_t^F = R_t + \frac{\tau^{1/2}(1 + \tau)^{1/2}[P_t - \alpha^G [E(g_t^1) - \bar{g}]] + 2\omega E(g_t^W) + 2(1 - \omega)E(g_t^I) - \alpha^G \varepsilon_t}{\tau^{1/2}(1 + \tau)^{1/2} - 2} \]  

(22)
It is straightforward to observe from (22) that the cost of serving public debt increases with positive realization of the GDP shock $\varepsilon_t$ in proportion to the weight $\alpha^G$ of the supply of these bonds by the Treasury. Incidentally, this hints at a further line of future research, namely to discuss what is the risk aversion by the public debt manager besides that of investors. In the central scenario where $\varepsilon_t = 0$, the predictable increase in the cost of debt borne out by the Treasury as a result of the issuance of GDP-Is is

$$C(\alpha^G) = \frac{\tau^{1/2}(1 + \tau)^{1/2}\{P_t - \alpha^G [E(g_1^t) - \bar{g}]\} + 2\omega E (g_1^W) + 2(1 - \omega)E(g_1^t)}{\tau^{1/2}(1 + \tau)^{1/2} - 2}$$

(23)

Even a simple visual inspection of result (23) suggests that, at least in principle, $C(\alpha^G)$ can take either sign. The denominator is positive under the usual mild condition $\tau > (\sqrt{17} - 1)/2$. As to the numerator, its sign is driven by the difference $\{P_t - \alpha^G [E(g_1^t) - \bar{g}]\}$ since this is pre-multiplied by the coefficient $\tau^{1/2}(1 + \tau)^{1/2}$ for which reasonable values could be in the order of 30, while the sum $[2\omega E (g_1^W) + 2(1 - \omega)E(g_1^t)]$ is about some percentage points. As to the term $\{P_t - \alpha^G [E(g_1^t) - \bar{g}]\}$, this can be positive but a negative result is not ruled out if $P_t$ is “small”, $\alpha^G$ is “large” and the Treasury sets the parameter $\bar{g}$ lower enough than $E(g_1^t)$. Let’s discuss more systematically these options through some numerical simulations.

Plot A of Chart 3 overleaf shows results of $C(\alpha^G)$ as a function of the weight $\alpha^G$ of GDP-Is out of the stock of public debt (values on the horizontal axis) and of the credit risk premium $P_t$, which is sampled at 0.5%, 2% and 3%. A first insight offered by this plot is that under suitable values of the parameters the cost turns negative, namely the Treasury would save off by issuing the GDP-Is. However, these values set challenging conditions: the credit-worthiness of the issuer needs to be fairly high with a spread over a risk-free asset of only 0.5% and the Treasury issues GDP-Is on a very large scale as the weight $\alpha^G$ must be not lower than 60%. Conversely, when this weight is in the order of 10-20% (which looks a more feasible target than 60%; see fn. 20), even an high-standing issuer would need to accept an increase in the average cost of the debt – of note: of the entire stock of debt, not only that financed by the GDP-Is – close to or higher than half of a percentage point. Actually, the additional cost soars and is always positive when the creditworthiness of the issuer is at best intermediate or lowish, as measured by a credit risk premium of 2% and 3%. For these issuers, the additional cost could be in the order of 2 or even more than 2 percentage points.

Plots B and C show the results of similar exercise, playing respectively on the foreign debt ownership $\omega$ and on the relative strength of domestic and foreign growth. All other things being equal, the additional cost is lower (albeit only marginally) when the weight $\omega$ is higher and it is also lower when domestic growth is lower than foreign growth (slightly less marginal difference). The real value added of plots B and C is gained in comparison with plot A, namely the magnitude of the additional cost may be quite variable but variability is almost entirely pinned down on only two parameters: $P_t$ and $\alpha^G$.

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21 Should the Treasury set $\bar{g}$ higher than $E(g_1^t)$, then the cost $C(\alpha^G)$ would be always positive.
Simulations of the cost function $C(\alpha^G)$

A. Sensitivity of cost to the credit risk premium
(Other parameters: $\tau = 30$, $E(g_t^I) = E(g_t^W) = 3\%$, $\bar{g} = 2\%$, $\omega = 30\%$)

B. Sensitivity of cost to the foreign debt ownership
(Other parameters: $\tau = 30$, $P_t = 2\%$, $E(g_t^I) = E(g_t^W) = 3\%$, $\bar{g} = 2\%$)

A. Sensitivity of cost to changes in domestic and foreign GDP
(Other parameters: $\tau = 30$, $P_t = 2\%$, $\bar{g} = 2\%$, $\omega = 30\%$)
5.1 The “broad” portfolio problem

Compared to the “narrow” portfolio problem, the “broad” problem looks more conventional since it involves the setting of the optimal weight $\alpha_F$ of a risk free asset where the alternative is a risky asset (the “narrow” portfolio). By arbitrage condition, the interest rate $r_t$ of this broad portfolio value is found as solution of:

$$r_t - r_t^F = (1 - \alpha_F)(r_t^a - r_t^F)$$

(24)

Imposing first order conditions on the Taylor expansion of a CRRA utility function yields the following long expression (details are in Annex 5)

$$2B_t + A_t \left\{ [-TB_t + \tau A_t + \tau E(g_B^t)] + \frac{2}{T-2} E(g_B^t) - \tau A_t \right\} \left[ TB_t - \frac{2}{T-2} E(g_B^t) \alpha_F \right]$$

$$- \tau(\tau+1)B_t \left\{ [A_t + E(g_B^t)] - A_t \alpha_F \right\}^2 = 0$$

(25a)

where

$$A_t \equiv P_t - \alpha^G [E(g_I^t) - \bar{g}] + \frac{T[P_t - \alpha^G [E(g_I^t) - \bar{g}]] + 2E(g_B^t)}{\tau^{1/2}(1+\tau)^{1/2}}$$

(25b)

$$B_t \equiv \frac{E(g_B^t)}{T-2}$$

(25c)

$$T \equiv \tau^{1/2}(1+\tau)^{1/2}$$

(25d)

while $P_t$ was defined in result (2). Expression (25a) is an equation to the second power in $\alpha_F$, whose analytical solutions while simple in concept imply a rather lengthy algebra (plus there is no obvious solution to the square root in the formula). In any case our interest here lies not much in the specific value of the solutions in $\alpha_F$ but rather on how these solutions change with $\alpha^G$. This can be done via a numerical procedure, results are shown in Table 3.

<table>
<thead>
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<th>$\alpha^G$</th>
<th>$\alpha^F$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5%</td>
<td>74.9%</td>
</tr>
<tr>
<td>10%</td>
<td>76.8%</td>
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<tr>
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<td>40%</td>
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</tr>
<tr>
<td>50%</td>
<td>95.5%</td>
</tr>
</tbody>
</table>

(1) Values shown in the right-hand column are solutions of equation (25a) for the values $\alpha^G$ shown in the left-hand column plus $E(g_I^t) = E(g_I^W) = 3\%$, $\bar{g} = 2\%$, $\omega = 30\%$, $\tau = 30$. 

Table 3

Values of the weight $\alpha_F$ of the broad portfolio as a function of the supply $\alpha^G$ of GDP-Is (1)
The results shown in this table suggest the weight $\alpha^F$ of the risk-free asset in the broad portfolio increases with the weight $\alpha^G$ of the GDP-Is out of the stock of (risky) government securities. The relationship between the two weights looks non-linear if one observes that when $\alpha^G$ goes from 10% to 20%, $\alpha^F$ increases by 3.9 percentage points, which become 4.4 points in the transition of $\alpha^G$ from 20% to 30% and then 4.9% and 5.5 points in respectively the transitions from 30% to 40% and from 40% to 50%. A possible economic intuition of the direct relationship established between the two weights would run as follows. The representative investor manages a given budget for risk, which is to be allocated across its portfolio. As explained above, the subscription of a security such as the GDP-I implies accepting a double risk (even for the foreign investor bar the case where $g^I_t$ and $g^W_t$ are uncorrelated) and this eats out a portion of the risk budget increasing with the weight $\alpha^G$. Accordingly, to balance this pattern, the investor is left with no other option than to buy more of the risk-free asset. Eventually, it is the conventional but yet risky government bond to feel the squeeze in demand by the investors. Thus, it seems reasonable to conjecture that the fall in demand brings about, in turn, an increase in the yield of the conventional bonds, all other things being equal.

Two remarks are here in order, though. Firstly, these are conjectures based on the results of a numerical exercise and thus depend on the selected values of individual parameters. As far as $E(g^I_t) > \bar{g}$, we could not get any sequence of values of the parameters for which $\alpha^F$ falls when $\alpha^G$ rises. However, we can’t rule out a priori that with enough trials eventually one such sequence might be obtained. Or, more easily, if one posits $E(g^I_t) < \bar{g}$, then say if $\alpha^G$ rises from 10% to 30% $\alpha^G$ decreases from 66.95% to 64.23%. However, besides the thoughts put forward in section 4.1 on why a Treasury could be lukewarm in setting a level of $\bar{g}$ distinctly higher than the consensus forecast on GDP, one should not lose sight of the fact that this indirect gain would be offset by the increase in direct costs related to GDP-Is according to function (23) describing $C(\alpha^G)$.

Secondly, the results gathered in this section as well as those obtained discussing $C(\alpha^G)$ apply under a ceteris paribus clause. In fact, and this is the subject of next section, this clause may not hold true because GDP-Is can lower the likelihood that $d_t > \bar{d}$. Accordingly, investors ought to seek a lower credit risk premium to underwrite the government bonds.

6. Changes in the probability of default

Let’s focus on the inequality within result (8b) defining the probability that $d_t > \bar{d}$ from which, after repeated substitutions, we derive the fuller but longer expression

$$v_t = 1 + 2E(g^I_t) \frac{d_{t-1} \epsilon_t}{[1 + E(g^I_t)]^2} + \frac{r^F_t + P_t + \frac{T}{(T-2)} \left[ E \left( P_t - \alpha^G [E(g^I_t) - \bar{g}] \right) + 2E(g^B_t) + \alpha^G \epsilon_t \right]}{\frac{1}{[1 + E(g^I_t)]^2} d_{t-1} \epsilon_t}$$

$$> \gamma + \frac{E(g^I_t)}{1 + E(g^I_t)} d_{t-1} - \frac{r^F_t + P_t + \frac{T}{(T-2)} \left[ E \left( P_t - \alpha^G [E(g^I_t) - \bar{g}] \right) + 2E(g^B_t) + \alpha^G \epsilon_t \right]}{1 + E(g^I_t)} d_{t-1}$$

(26)

---

22 We obtain these results imposing $\bar{g} = 4\%$, and using for the other parameters the values specified in table 3.
Next, let’s examine the two shocks in turn. Starting with the pair \( \{ \nu_t > 0; \varepsilon_t = 0 \} \), (26) becomes

\[
v_t > \gamma + \left\{ \frac{E(g_t^l) - r_t^F - p_t - \frac{T P_t + 2 E(g_t^B)}{(T - 2)}}{1 + E(g_t^l)} + \frac{T \alpha^G [E(g_t^l) - \bar{g}]}{(T - 2)} \right\} d_{t-1} \tag{27a}
\]

It is straightforward to observe that a positive value of \( \alpha^G \) plays a role in (27a) only subject to \( E(g_t^l) \neq \bar{g} \). Noticeably, if \( E(g_t^l) > \bar{g} \), the right-hand side of (27a) rises with \( \alpha^G \), reducing the odds that for a given realization of the debt shock \( \nu_t \) the inequality is verified and with it the probability that \( d_t > \bar{d} \). Namely if the Treasury means to use GDP-Is to mitigate the transfer of debt shocks on higher levels of the probability of default, then it would be advised to set the reference level \( \bar{g} \) below the consensus expectation of the GDP change. To get an idea of the magnitude in the change of the right-hand side of (27a) that the Treasury could engineer by playing on \( \bar{g} \), in a numerical exercise with \( E(g_t^l) = E(g_t^B) = 3\% \), \( \tau = 30 \), \( P_t = 1\% \), \( \alpha^G = 30\% \) we work out an increase by 0.31 percentage points for each percentage point of difference between \( E(g_t^l) \) and \( \bar{g} \). While it is beyond the scope of this paper an empirical research on the range of values that a debt-shock \( \nu_t \) could take, an educated guess is that the variability of these shocks could far exceed any order of dimension measured in a few tenths of percentage point. It could be enough to observe that in Ireland at the peak of the banking crisis, the increase in the debt ratio owing to the bailing in of the banking system totaled well in excess of a dozen of percentage point of GDP. To cut a long story short, GDP-Is could play at best only a marginal role in mitigating the translation of debt shock on the probability of default.

More complex is the algebra in the alternative pair \( \{ \nu_t = 0; \varepsilon_t > 0 \} \). Here, (26) becomes

\[
\left\{ \frac{1 + 2 E(g_t^l)}{[1 + E(g_t^l)]^2} + \frac{r_t^F + P_t + T \{ P_t - \alpha^G [E(g_t^l) - \bar{g}] \} + 2 E(g_t^B)}{(T - 2)} \right\} d_{t-1} \varepsilon_t + \frac{\alpha^G}{[1 + E(g_t^l)]^2} d_{t-1} \varepsilon_t^2 > \gamma + \frac{E(g_t^l) - r_t^F - P_t - \frac{T \{ P_t - \alpha^G [E(g_t^l) - \bar{g}] \} + 2 E(g_t^B)}{(T - 2)}}{1 + E(g_t^l)} d_{t-1} \varepsilon_t^2 \tag{27b}
\]

A first observation about result (27b) is that here a non-zero value of \( \alpha^G \) brings about an impact on the left-hand side of the inequality even for \( E(g_t^l) = \bar{g} \). This defines the role GDP-Is may play on stymieing GDP shocks quite distinctly from the conclusions we have just reached on their role with respect to debt shocks.

Second, focusing only on the terms with \( \alpha^G \), the change in the difference between the left hand-side and the right-hand-side of (3) for given level of \( \alpha^G \) is

\[
\Delta_{\alpha^G} = d_{t-1} \left\{ - \frac{T [E(g_t^l) - \bar{g}]}{(T - 2) [1 + E(g_t^l)]^2} (\varepsilon_t + 1) + \frac{1}{1 + E(g_t^l)} \varepsilon_t + \frac{1}{[1 + E(g_t^l)]^2} \varepsilon_t^2 \right\} \tag{28a}
\]
Note that the coefficients before the term \((\varepsilon_t + 1)\) is way lower (in relative and absolute value) than the one before the term \(\varepsilon_t\). Say, if \(E(g^1_t) = 4\%\), \(\bar{g} = 3\%\) and \(\tau = 30\) then these two coefficients are respectively -0.0099 and 0.9615, that is they are different by a factor of almost 100. As to the term with \(\varepsilon_t^2\), the factor of proportionality gets up roughly 1.000 when one counts for both the coefficient and the term itself. Thus, we can safely gain an intuition by focusing only on the term in \(\varepsilon_t\)

\[
\Delta \alpha^G \propto \frac{1}{1 + E(g^1_t)} \ d_{t-1} \ \varepsilon_t
\] (28b)

It follows that faced with a negative shock \(\varepsilon_t\) (growth lower than expected), given a positive \(\alpha^G\) the Treasury benefits from a reduction in the probability of default which increases with the (absolute value) shock itself, the level of the debt ratio prior to the shock while it decreases with the expected change in nominal GDP. From this perspective GDP-I bonds are effective in mitigating the transfer of a growth shock on the probability of default.

To get a more complete picture and to understand how large this gain can be, we work out a total of 240 result (27b), trying out 5 values for \(\alpha^G\) (0\%, 10\%, 30\%, 50\% and 70\%), 2 for P (1\% and 2\%), 3 for \(\varepsilon_t\) (-1\%, -3\% and -5\%), 2 for the weight \(\omega\) of debt ownership by foreign investors (30\% and 50\%) and finally 2 each for \(E(g^W_t)\) and \(E(g^I_t)\). As to the other parameter, \(\bar{g} = E(g^I_t) - 1\%, \tau_t^F = 1\%, \tau = 30\). We report the results of the exercise using two statistics: the number of instances (out of 48 trials per each value of \(\alpha^G\)) when the condition \(d_t > \bar{d}\) provided that \(d_{t-1} > \bar{d}\); the average value of the ratio \(d_t / \bar{d}\) in trials where \(d_{t-1} > \bar{d}\). Namely the first statistics counts the number of defaults and the second one measures how close we got near that event without (fortunately) actually having it. That said, some contribution from the recourse to GDP-Is can bring about material gains in the mitigation of GDP shocks if the share of these bonds is at least in the order of 30\%. Even at that level, we observe a decline of the ratio of the second statistics to 39\% from 52\% in a world without GDP-Is; however, even with these bonds the number of default is still counted equal to 4, the same number of events of default we identified in the scenario without GDP-Is \((\alpha^G = 0\%)\). If the share \(\alpha^G\) rises to 50\%, the number of defaults finally falls to 0 and if the weight is 70\% the average ratio falls further by one third.

<table>
<thead>
<tr>
<th>Weight (\alpha^G) of recourse to GDP-Is</th>
<th>Number of times (d_t &gt; \bar{d})</th>
<th>Average ratio (d_t / \bar{d}), when (d_t &lt; \bar{d})</th>
</tr>
</thead>
<tbody>
<tr>
<td>0%</td>
<td>4</td>
<td>52%</td>
</tr>
<tr>
<td>10%</td>
<td>4</td>
<td>48%</td>
</tr>
<tr>
<td>30%</td>
<td>4</td>
<td>39%</td>
</tr>
<tr>
<td>50%</td>
<td>0</td>
<td>35%</td>
</tr>
<tr>
<td>70%</td>
<td>0</td>
<td>23%</td>
</tr>
</tbody>
</table>

(1) Each row reports results of 48 simulations over equation (28b), based on combinations of assumptions on \(\alpha^G\) (0\%, 10\%, 30\%, 50\% and 70\%), \(P\) (1\% and 2\%), \(\varepsilon_t\) (-1\%, -3\% and -5\%), \(\omega\) (30\% and 50\%) \(E(g^W_t)\) (3\% and 5\%) and \(E(g^I_t)\) (3\% and 5\%). As to the other parameter, \(\bar{g} = E(g^I_t) - 1\%, \tau_t^F = 1\%, \tau = 30\).
We conclude this section by discussing the topic of perception by investors that the State issuing the GDP-Is might manipulate official statistics to reduce the interest rate bill. A way to approach analytically the implications of such perception, should this become entrenched in investors’ beliefs, is first by taking expectations of (26). Recalling the assumptions \( E(v_t) = 0, \ Var(v_t) = \sigma^2_v, E(\varepsilon_t) = 0, \ Var(\varepsilon_t) = \sigma^2_{\varepsilon} \), one obtains

\[
\frac{\alpha^G}{[1 + E(g^B_t)]^2} \sigma^2_{\varepsilon} > \gamma [1 + E(g^B_t)] + \left\{ E(g^B_t) - r^B_P - \frac{T \left[ R - \alpha^G \left[ E(g^B_t) - \bar{g} \right] + 2 \ E \left( g^B_t \right) \right]}{T - 2} \right\} d_{t-1}
\]

(29)

Then, one way to figure out the impact of the said perception is to posit that investors could raise the estimated variance \( \sigma^2_{\varepsilon} \) fearing more uncertainty in the GDP change should official statistics be reckoned to be unreliable. Then, by itself this market belief would raise the left-hand side of (29) while it would not have any bearing on the right-hand side. That is, a higher \( \sigma^2_{\varepsilon} \) translates into a higher probability that \( d_t > \bar{d} \) and thus a higher risk premium. Namely a Treasury could be punished by the market ex ante. Whether this change in the probability of default is material depends, perhaps unsurprisingly, on the weight \( \alpha^G \) of the recourse to GDP-Is and on the debt-to-GDP ratio as of time \( t-1 \) (before the market belief gets established).

7. Concluding remarks

In the language of mathematics a result is understood to be “correct” if it is derived consistently and without flaws from a set of starting assumptions. In this sense, the ambition of this paper is to present “consistent” results on the implications of issuing GDP-Indexed bonds, along more conventional ones, to finance public debt elaborating on four basic relations: one describes the dynamics of the debt-to-GDP ratio; a second one breaks down the interest rate on a sovereign bond as the sum of a return on a risk free asset plus a premium on credit risk (to which later, we add a premium on the uncertainty on the outturn in GDP itself); a third relation simply states that any forecast of GDP is subject to errors; the fourth and final relation describes a reaction function pursued by the policy maker to steer the primary surplus, subject to a ceiling reflecting the “fiscal fatigue”.

These four basic relations, plus a definition of probability of default, yield a surprising richness of results which may shed light on the way the recourse to GDP-Is would affect the cost of financing public debt but also the extent these securities would dilute the impact of different types of shocks on the probability of default of the Treasury.

Compared to the extant and rapidly increasing literature on GDP-I, to the best of our knowledge this paper introduces two elements of novelty. Firstly, the argument of the investor’s CRRA utility function, through which the portfolio allocation is solved, includes the return on government securities but also, and this is the novelty, a measure of non-financial income. In this way we wish to study how co-movements in the two sources of income – co-movements which are especially relevant when the investor is resident in the country issuing the securities but play some role for
foreign investors too – affect the equilibrium yield at which the market would underwrite the GDP-Is.

As a second element of novelty, the portfolio problem is broadened to include, besides the customary choice between GDP-Is and nominal bonds issued by a given Treasury, also a third (risk free) asset issued by another party. By doing so, the approach we put forward does not constrain the overall demand for the securities issued by the Treasury to be constant. Rather, we accept that as a consequence of the issuance of GDP-Is, the overall demand for the government securities may undergo shifts and, as a result, the equilibrium yield may vary on nominal bonds too. To see how this could work, on the one hand, if GDP-Is deliver what proponents see in them – a tool to avoid that an unforeseen recession triggers a crisis in public debt financing – then investors ought to reckon a decrease in the credit risk, seeking a lower remuneration on this risk across the board. On the other hand, investors who purchase GDP-Is are taking on additional risk (risk on GDP outturn on top of credit risk) and ceteris paribus that may eat out a greater amount of their budget risk. The ultimate result is to lower the demand for all risky assets, including the nominal bonds.

By elaborating on the algebraic set-up defined by the four aforementioned equations and through numerical simulations, we obtain the following main results.

Firstly, compared to a baseline scenario in which the Treasury relies only on conventional bonds, the additional cost of financing the public debt incurred by mixing conventional and GDP-I bonds is markedly heterogeneous across economic scenarios, being dependent on a number of parameters. Among them, the most important ones are the credit risk premium required by investors in the baseline scenario and the share of debt financed through these securities. Some role plays also the share of debt held by foreign investors and the relative dimension of the change in domestic and ‘rest-of-the-world’ GDP. It follows that the additional cost is more accurately expressed in terms of a range than a single-point estimate: according to simulations, the issuance of GDP-Is could increase the cost of financing debt by one, two or even three percentage points under most scenarios. However, the additional cost may even turn negative (namely there would be a saving), provided the Treasury relies massively on the GDP-Is, for a share of 60% of more of public debt, and its creditworthiness is fairly high. One could also note in this respect that if creditworthiness is high, the Treasury ought to be sheltered from the risk of default due to low growth. Namely the Treasury could have limited scope for venturing in a new asset class.

Second, GDP-Is would actually do what is expected from them, namely to loosen the impact of GDP shocks (growth lower than expected) on the probability of default, preventing the development of a recession into a crisis of public finances. However, material gains in this respect require the Treasury to issue GDP-Is heavily, at least in the proportion of at least 30% of the overall stock of debt while more meaningfully gains would need for a proportion in the order of 50%.

Third, on top of the direct cost mentioned above, there would be also indirect sources of cost for the Treasury, of opposite signs. On the one hand, as a result of the issuance of GDP-Is, investors would lower their demand for the government securities across the board, and prominently the demand for conventional bonds, and this should bring about a rise in yields, ceteris paribus. On the other hand, as GDP-Is may mitigate the impact of GDP shocks on the probability of default, it is reasonable to
expect that investors will seek a lower remuneration on account of the credit risk, lowering yields (again, all other things being equal) charged on government bonds.

Putting all these elements together and as it should be expected from a state contingent bond, GDP-Is are a good medicine for the type of illness to which they are addressed – mitigating the impact of an unforeseen recession on public finances – but are not a remedy for all problems. Plus, they may be rather costly. And when they are not, it is because the Treasury is hardly at risk of default in the first place.
1. The parameter $\beta$ in the fiscal rule (4a)

Replacing (4a) in (1b) when the ceiling $\gamma$ is not binding, one has

\[
d_t = \frac{(1 - \beta) r_t + 1 + \beta g_t^I}{1 + g_t^I} \, d_{t-1} + v_t
\]

where

- if $\beta = 1$, \( d_t = d_{t-1} + v_t \implies \frac{\partial d_t}{\partial g_t^I} = 0 \)
- if $\beta \neq 1$,

\[
\frac{\partial d_t}{\partial g_t^I} = \frac{(\beta - 1)(1 + r_t)}{(1 + g_t^I)^2}
\]

2. Result on the probability of default

In the main text we derive result (8b) from (8a) provided that

\[
r_t - \frac{g_t^I}{1 + g_t^I} \, d_{t-1} \equiv \frac{1 - r_t + 2E(g_t)}{\left(1 + E(g_t)\right)^2} \, d_{t-1} \, \varepsilon_t + \frac{r_t - E(g_t)}{1 + E(g_t)} \, d_{t-1}
\]

(\text{A.2})

What follows is a proof that the approximation in (A.2) holds under relatively mild conditions. By substitution of result (3a) of the main text in the term on the left-hand side of (A.2)

\[
r_t - \frac{g_t^I}{1 + g_t^I} \, d_{t-1} = \frac{r_t - E(g_t)}{1 + E(g_t) + \varepsilon_t} - \varepsilon_t
\]

(A.3)

Let’s rewrite the right-hand side of (A.3) as

\[
\frac{x_1 - \varepsilon_t}{x_2 + \varepsilon_t} \, d_{t-1} = r_t \, \varepsilon_t + y_t
\]

(A.4)

where $r_t$ and $y_t$ are expressions to be determined and $x_1 \equiv r_t - E(g_t)$ and $x_2 \equiv 1 + E(g_t)$ are new symbols introduced to shorten notation. Multiplying both sides of (A.4) by $(x_2 + \varepsilon_t)$

\[
x_1 \, d_{t-1} - \varepsilon_t \, d_{t-1} = r_t \, x_2 \, \varepsilon_t + \beta \, y_t + r_t \, (\varepsilon_t)^2 + y_t \, \varepsilon_t
\]

Next, we neglect the term in the second power of $\varepsilon_t$ – and this is the approximation involved in (A.2) which should be small compared to the terms to the first power so long as the shock is in the order of some percentage points – to derive

\[
x_1 \, d_{t-1} - \varepsilon_t \, d_{t-1} \approx x_2 \, y_t + (r_t \, x_2 + y_t) \, \varepsilon_t
\]

(A.5)

To find suitable expressions for $r_t$ and $y_t$ we imposed equality in the coefficients of the terms without $\varepsilon_t$ on the two sides of (A.5) and then doing the same for the coefficients of the terms with $\varepsilon_t$, we have

\[
\begin{cases}
    x_1 \, d_{t-1} = x_2 \, y_t \\
- d_{t-1} = r_t x_2 + y_t \\
    y_t = \frac{r_t - E(g_t)}{1 + E(g_t)} \, d_{t-1} \\
    r_t = - \frac{1 - r_t + 2E(g_t)}{[1 + E(g_t)]^2} \, d_{t-1}
\end{cases}
\]

3. Formulae on indexation

In a GDP-I contract where indexation applies both to the coupon and the principal, the following holds

\[
1 + r_t^N = (1 + g_t^I) + r_t^{g,p}
\]

(A.6)
from which given result (9b) of the main text, it is straightforward to derive result (10a).
In a scenario where the coupon is paid not only at maturity, the cash flow structures looks like
(under the simplest alternative of coupon paid at maturity and at an intermediate time)

\[
\begin{array}{ccc}
\text{nominal bond} & t_0 & t_1 & t_2 \\
\text{GDP-I (only coupon is indexed)} & -1 & r_{t1}^G & 1 + r_{t2}^G \\
\text{(both coupon and principal)} & -1 & r_{t1}^G & (1 + g_{t1}^l + g_{t2}^l) + r_{t2}^p \\
\end{array}
\]

where

\begin{align*}
& r_{t1}^G \text{ interest rate paid ex post after the first period on the GDP-I} \\
& r_{t2}^G \text{ interest rate paid ex post after the second period on the GDP-I} \\
& g_{t1}^l, g_{t2}^l \text{ rates of change in nominal GDP in the first and second period respectively} \\
\end{align*}

As we are comparing cash flows in different periods, we need to introduce also a factor of capitalization which we denote \( \rho \). Hence, in the scenario where indexation applies only to the coupon the arbitrage condition is

\[
r_t^N (1 + \rho) + (1 + r_t^N) = r_{t1}^G (1 + \rho) + (1 + r_{t2}^G) \tag{A.7}
\]

Taking expectations one has

\[
r_t^N (2 + \rho) + 1 = (1 + \rho) E_t (r_{t1}^G) + 1 + E_t (r_{t2}^G) \\
\Rightarrow \quad r_t^{G,a} = r_t^N - [E_t (g_{t1}^l) + E_t (g_{t2}^l)] (2 + \rho)
\]

which is result (10b) of the main text.

Finally, under the scenario where indexation applies to both coupon and principal and the coupon is paid more than once, the arbitrage condition is

\[
z_t^N (1 + \rho) + (1 + z_t^N) = z_{t1}^G (1 + \rho) + (1 + g_{t1}^l + g_{t2}^l) + z_{t2}^{G,a} \tag{A.8}
\]

Taking expectations one has

\[
r_t^N (2 + \rho) + 1 = (1 + \rho) E_t (r_{t1}^G) + 1 + E_t (g_{t1}^l) + E_t (g_{t2}^l) + E_t (r_{t2}^G) \\
\Rightarrow \quad r_t^{G,a} = r_t^N - [E_t (g_{t1}^l g_{t1}) + E_t (g_{t2}^l) + E_t (r_{t2}^G)] (2 + \rho)
\]

which is result (10c).

4. The result for \( \hat{\eta} \)

For \( c_1 > 0 \) and \( c_1 \neq 1 \), we write the CRRA utility function as

\[
U(c_1) = \frac{1}{1 - \tau} [r_t^a - r_t^F + E(g_t^B)]^{1 - \tau} \\
\]

where the first and second derivatives are respectively

\[
U'(c_1) = [r_t^a - r_t^F + E(g_t^B)]^{-\tau} \quad U''(c_1) = -\tau [r_t^a - r_t^F + E(g_t^B)]^{-\tau - 1}
\]

Recalling the general expression of the Taylor expansion up to the second order

\[
U(x) \cong U(0) + U'(0) x + \frac{1}{2} U''(0) x^2
\]

where we set as zero-point the sum \( r_t^a - r_t^F + E(g_t^B) \) at \( p_t = 0 \) and \( \delta = 0 \):

\[
U[r_t^a - r_t^F + E(g_t^B); p_t = 0, \delta = 0] = \alpha^G \eta \tau \sigma^2_t + E(g_t^B) \tag{A.9}
\]

Hence,
\[
U[\alpha^G \eta \tau \sigma^2_a + E(g^B_t)] \approx \frac{1}{1 - \tau} \{x_1 \eta + E(g^B_t)\}^{-1 - \tau} \{\{x_1 \eta + E(g^B_t)\}^2 + (1 - \tau)\{x_1 \eta + E(g^B_t)\}(x_2 + x_1 \eta) - \frac{1}{2} \tau (1 - \tau) (x_2 + x_1 \eta)^2\} \tag{A.10}
\]

where to keep the notation more compact we used 
\[
x_1 = \alpha^G \tau \sigma^2_a > 0
\]
\[
x_2 = \frac{p_t}{1 - p_t} (1 - \theta)(1 + r^F_t) - \alpha^G E(g^I_t)
\]\n\[
r^a_t - r^F_t = \frac{p_t}{1 - p_t} (1 - \theta)(1 + r^F_t) - \alpha^G E(g^I_t) + \alpha^G \eta \tau \sigma^2_a = x_2 + x_1 \eta
\]

Taking the derivative of (A.10) w.r.t. \( \eta \) and imposing first order conditions

\[
- \frac{1 + \tau}{1 - \tau} [x_1 \eta + E(g^B_t)]^{-2 - \tau} x_1 \left\{\{x_1 \eta + E(g^B_t)\}^2 + (1 - \tau)\{x_1 \eta + E(g^B_t)\}(x_2 + x_1 \eta) - \frac{1}{2} \tau (1 - \tau) (x_2 + x_1 \eta)^2\right\}
\]
\[
= - \frac{1}{1 - \tau} [x_1 \eta + E(g^B_t)]^{-1 - \tau} \left\{2[x_1 \eta + E(g^B_t)]x_1 + (1 - \tau)x_1(x_2 + x_1 \eta) + (1 - \tau)[x_1 \eta + E(g^B_t)]x_1 - \frac{1}{2} \tau (1 - \tau) 2(x_2 + x_1 \eta)x_1\right\}
\]

\[
[4 + \tau (1 + \tau) - 4\tau] x_1^2 \eta^2 + \{8 x_1 E(g^B_t) + 2\tau (1 + \tau)x_1 x_2 - 4\tau x_1 x_2 - 4\tau x_1 E(g^B_t)\} \eta + 4\left(E(g^B_t)^2 + \tau (1 + \tau)x_2^2 - 4\tau x_2 E(g^B_t) = 0 \right. \tag{A.11}
\]

which is approximately (see below on the magnitude of the approximation)

\[
\left\{[2 - \tau^{1/2}(1 + \tau)^{1/2}] x_1 \eta + [2E(g^B_t) + \tau^{1/2}(1 + \tau)^{1/2}x_2]\right\}^2 \approx 0 \tag{A.12}
\]

Hence

\[
\hat{\eta} = \frac{\tau^{1/2}(1 + \tau)^{1/2}x_2 + 2E(g^B_t)}{[\tau^{1/2}(1 + \tau)^{1/2} - 2]x_1} \tag{A.13}
\]

Replacing back the expressions for \( x_1 \) and \( x_2 \) one finds

\[
\hat{\eta} = \frac{\tau^{1/2}(1 + \tau)^{1/2} \left[\frac{p_t}{1 - p_t} (1 - \theta)(1 + r^F_t) - \alpha^G E(g^I_t)\right] + 2E(g^B_t)}{[\tau^{1/2}(1 + \tau)^{1/2} - 2]\alpha^G \tau \sigma^2_a}
\]

which is result (16a) of the main text.

The approximation engineered to derive (A.11) can be worked out as follows

\[
[2 - \tau^{1/2}(1 + \tau)^{1/2}] x_1 \eta + [2E(g^B_t) + \tau^{1/2}(1 + \tau)^{1/2}x_2]^2 - \{4 + \tau (1 + \tau) - 4\tau x_1^2 \eta^2\}
\]
\[
- \{8 E(g^B_t) + 2\tau (1 + \tau)x_1 x_2 - 4\tau x_1 x_2 - 4\tau E(g^B_t)\} x_1 \eta
\]
\[
- \{4 + \tau (1 + \tau) - 4\tau x_1^2 \eta^2 + [8 x_1 E(g^B_t) + 2\tau (1 + \tau)x_1 x_2 - 4\tau x_1 x_2 - 4\tau x_1 E(g^B_t)\} \eta + 4\left(E(g^B_t)^2 + \tau (1 + \tau)x_2^2 - 4\tau x_2 E(g^B_t)\right) = K
\]

\[
4\{[\tau - \tau^{1/2}(1 + \tau)^{1/2}] x_1^2 \eta^2 + [\tau + \tau^{1/2}(1 + \tau)^{1/2}]x_2 E(g^B_t)
\]
\[
+ \left[\tau x_2 + \tau E(g^B_t) + \tau^{1/2}(1 + \tau)^{1/2}x_2 - \tau^{1/2}(1 + \tau)^{1/2}E(g^B_t)\]x_1 \eta\}
\]

\[
= K \tag{A.14}
\]
For $\tau = 30$, following Janecek (2004), the coefficient before $x_1^2\eta^2$ in (A.14) is $-1.98$ while the coefficient before the same term in (A.11) is $812.02$. Hence the correction $K$ can be considered as small.

5. Derivation of the solution for $\alpha^F$

We use here: the arbitrage result (24) of the main text; a CRRA utility function with argument

$$c_2 = r_t^B - r_t^F + E(g_t^B) = (1 - \alpha^F)A_t + E(g_t^B) \quad (A.15)$$

$$A_t \equiv P_t - \alpha^G [E(g_t^B) - \bar{g}] + \alpha^G \bar{\eta} \tau \sigma_t^2 \quad (A.16)$$

We set as zero-point the sum $r_t - r_t^F + E(g_t^B)$ at $p_t = 0$ and $E(g_t^B) = \bar{g}$

$$[r_t^B - r_t^F + E(g_t^B)]|_{p_t=0; E(g_t^B) = \bar{g}} = \frac{\tau^{1/2}(1 + \tau)^{1/2} - 2\alpha^F}{\tau^{1/2}(1 + \tau)^{1/2} - 2}E(g_t^B) \quad (A.17)$$

Hence, in the Taylor approximation up to the second order

$$U[r_t^B - r_t^F + E(g_t^B)]$$

$$= \frac{1}{1 - \tau} \left[ \frac{T - 2\alpha^F E(g_t^B)}{T - 2} \right]^{1-\tau} + \left[ \frac{T - 2\alpha^F E(g_t^B)}{T - 2} \right]^{-1} \left[ (1 - \alpha^F)A_t + E(g_t^B) \right]$$

$$- \frac{1}{2} \tau \left[ \frac{T - 2\alpha^F E(g_t^B)}{T - 2} \right]^{-1} \left[ (1 - \alpha^F)A_t + E(g_t^B) \right]^2 \quad (A.18)$$

$$T \equiv \tau^{1/2}(1 + \tau)^{1/2} \quad (A.19)$$

Taking the derivative of (A.18) w.r.t. the unknown parameter $\alpha^F$ and imposing the result equal to 0, one has

$$\left[ \frac{T - 2\alpha^F E(g_t^B)}{T - 2} \right]^{-\tau} (-2) \frac{E(g_t^B)}{T - 2} - \tau \left[ \frac{T - 2\alpha^F E(g_t^B)}{T - 2} \right]^{-\tau-1} \left[ (1 - \alpha^F)A_t + E(g_t^B) \right]$$

$$+ \left[ \frac{T - 2\alpha^F E(g_t^B)}{T - 2} \right]^{-\tau} (-1)A_t$$

$$- \frac{1}{2} \tau \left[ \frac{T - 2\alpha^F E(g_t^B)}{T - 2} \right]^{-2} (-\tau - 1)(-2) \frac{E(g_t^B)}{T - 2} \left[ (1 - \alpha^F)A_t + E(g_t^B) \right]^2$$

$$- \frac{1}{2} \tau \left[ \frac{T - 2\alpha^F E(g_t^B)}{T - 2} \right]^{-1} 2\left[ (1 - \alpha^F)A_t + E(g_t^B) \right] (-1)A_t = 0$$

After some steps of sheer calculus and introducing the additional symbol

$$B_t \equiv \frac{E(g_t^B)}{T - 2} \quad (A.20)$$

One obtains

$$(2B_t + A_t) \left\{ -T B_t + \frac{2}{T - 2} E(g_t^B) \alpha^F + \tau (1 - \alpha^F)A_t + \tau E(g_t^B) \right\} \left[ T B_t - \frac{2}{T - 2} E(g_t^B) \alpha^F \right]$$

$$- \tau (\tau + 1) B_t \left[ (1 - \alpha^F)A_t + E(g_t^B) \right]^2 = 0$$

Which is result (25b) of the main text.
REFERENCES

Janecek K. (2004). What is a realistic aversion to risk for real-world individual investors?