



**MASSIMO BERNASCHI,**  
IAC - CNR

**ROBERTO MOREA,**  
SOGEI

**LUCIO SARNO,**  
Cambridge Judge Business School,  
University of Cambridge

**FABRIZIO TESSERI,**  
MEF - Department of Treasury

**FEDERICA VERANI,**  
MEF - Department of Treasury

**DAVIDE VERGNI,**  
IAC - CNR

# An Integrated Approach to Cost-Risk Analysis in Public Debt Management

# Index

<b>Abstract .....</b>	<b>2</b>
<b>Introduction .....</b>	<b>3</b>
<b>Mathematical Model .....</b>	<b>4</b>
<b>Interest rate scenarios .....</b>	<b>4</b>
Modelling the dynamics of interest rates .....	5
Anchoring the yield curves using exogenous information on short term rates .....	9
The debt portfolio cash flow and the balance constraint .....	11
The ESA 2010 cost function .....	12
Analytical study on the ESA 2010 evolution .....	13
<b>The Cost/Risk Analysis .....</b>	<b>14</b>
Classical measures of cost and risk .....	15
Evolution measures of cost and risk .....	16
<b>CVA modelling .....</b>	<b>17</b>
<b>Software Architecture .....</b>	<b>20</b>
<b>Conclusions .....</b>	<b>22</b>
<b>Appendix .....</b>	<b>22</b>
Generation of interest rate curves .....	22
<b>References .....</b>	<b>27</b>

## An Integrated Approach to Cost-Risk Analysis in Public Debt Management

Massimo Bernaschi<sup>1,2</sup> Roberto Morea<sup>3</sup> Lucio Sarno<sup>2,4</sup>  
Fabrizio Tesseri<sup>2</sup> Federica Verani<sup>2</sup> Davide Vergni<sup>1,2</sup>

### Abstract

*Public debt management requires accurate analysis of the cost and the risk associated with feasible choices of public debt portfolios. The Italian Treasury employs an in-house, custom-built set of models and software tools aimed at selecting portfolios of securities that satisfy the government's borrowing needs while fulfilling all the relevant regulatory constraints, thereby delivering suitable tradeoffs between cost and risk on an efficient frontier. In this paper, we present a description of the approach followed by the Italian Treasury, including the study of the dynamics of the cash flow generated by the public debt and the development of stochastic models for the evolution of the main risk factors (i.e., interest rates and inflation). Then, we describe how several dynamic metrics of cost and risks are integrated within a scenario generator in a modular software package called SAPE (Sistema Analisi Portafogli di Emissione – System for the Analysis of Public Debt Issuances). This system supports the public debt manager by providing accurate quantitative estimates of the expected effects of their choices taking into account not only shocks to interest rate curves but also exogenous forecasts about the future behavior of the risk factors. The software can also be used as an accounting tool for the outstanding securities of the Italian public debt, and includes various satellite modules for the evaluation of other relevant metrics, such as the Credit Value Adjustment for derivative instruments.*

*The latest version of the paper has been presented during the September 2019 "Public Debt Management Network Conference" organized in Paris by the PDM - Public Debt Management - Network, an initiative fostered by the OECD, the Italian Treasury and the World Bank to disseminate Public Debt management techniques; it will be published into the proceedings of the conference.*

<sup>1</sup> Istituto per le Applicazioni del Calcolo "M. Picone" (IAC) - CNR

<sup>2</sup> Dipartimento del Tesoro - Ministero dell'Economia e delle Finanze

<sup>3</sup> Sogei

<sup>4</sup> Cambridge Judge Business School, University of Cambridge

## Introduction

Public debt management is the process of designing and executing a strategy for the management of government debt with the ultimate goal to satisfy the financing needs of the government. In this process it is necessary to consider a number of relevant issues, related to cost, risk, maturity structure, and choice of instruments from a broad universe of assets. Debt managers generally rely on a cost-risk analysis of their stock of liabilities to make choices of debt instruments and portfolio composition.

This kind of analysis requires a broad set of tools. A key component of the toolkit required is a formal framework that allows to produce stochastic simulations of multiple term structures of interest rates and macroeconomic variables. From these simulations, which constitute a set of possible future scenarios for the interest rates' evolution, debt managers can determine the relationship between cost and risk they face under different feasible portfolios. The process is, in practice, made much more complex by a number of exogenous factors (i.e., factors not under the control of debt managers) that include, for example, the influence of central banks on the path of interest rates, political risk, global risks of various form, and regulatory constraints. Surprisingly, the literature in this area is scant, with few papers attempting to describe the modelling strategies and challenges faced in the process of evaluation of the cost and risk trade-off associated with different financing strategies. Stochastic simulation modelling is used in public debt decision making by many governments around the world. Just to cite a few we mention Canada (see Bolder 2002, 2003, 2011), the Swedish National Debt Office (see Bergstrom, Holmlund and Lindberg 2002), the National Bank of Denmark (2005), the UK Debt Management Office (see Pick and Anthony 2006), the French Treasury (see Renne and Sagne 2008), the Turkish Treasury (see Balibek and Memis 2012) and the U.S. Treasury (see Belton et al. 2018). For an overview of stochastic debt strategy simulation modelling in OECD countries, see Risbjerg and Holmlund (2005) while a general discussion on debt strategies in Europe can be found in Balling, Gnan and Holler (2013). There are also more academic works on public debt management that can be considered as food for thought for debt managers, see for example Dottori and Manna (2016). Finally, concerning practical applications, mathematical methods on cash flow forecasting and cost-risk analysis for public debt has been endorsed by the World Bank, which has developed a toolkit specifically designed by its debt and risk management experts (see Panzer 2015).

The Italian Treasury employs an in-house, custom-built set models and software tools aimed at selecting portfolios of securities that satisfy the government's borrowing needs while fulfilling all the relevant regulatory constraints. In this paper, we present a description of the approach followed by the Italian Treasury, including the study of the dynamics of the cash flow generated by the public debt and the development of stochastic models for the evolution of the main risk factors (i.e., interest rates and inflation). Then, we describe how several dynamic metrics of cost and risks are integrated within a scenario generator in a modular software package called SAPE (Sistema Analisi Portafogli di Emissione – System for the Analysis of Public Debt Issuances). This system supports the public debt manager by providing accurate quantitative estimates of the expected effects of their choices taking into account not only shocks to interest rate curves but also exogenous forecasts about the future behavior of the risk factors. The software can also be used as an accounting tool for the outstanding securities of the Italian public debt, and includes various satellite modules for the evaluation of other relevant metrics, such as the Credit Value Adjustment for derivative instruments. We provide both a description of the general framework and several illustrations of the way this set of tools is utilized in practice.

## Mathematical Model

The study of the evolution of the public debt and the assessment of the cost and risk of a portfolio of government securities with respect to a set of scenarios of the future interest rates, require a setup that represents in a quantitative way all the elements that contribute to the cash flows of all the relevant securities in the portfolio. The Italian Treasury Department issues six different types of domestic securities including three with a floating rate and a small number of foreign securities for a total amount of 2,004 billion euro plus a derivative portfolio with a notional of 99 billion euro at December 2019. Each type of security can be issued with different maturities and the total number of different securities is about twenty (see Table 1).

Securities	Amount (mln €)	%	Maturities (months)	Coupon	Indexed
BOT	113,929	5.68	6, 12	Zero Coupon	
CCTeu	125,586	6.26	60, 84	Floating Coupon	Euribor 6M
CTZ	51,139	2.55	24	Zero Coupon	
BTP	1,440,016	71.83	36, 60, ... , 360, 600	Fixed Coupon	
BTPs €i	151,707	7.57	60, 120, 180, 360	Floating Coupon	HIPC
BTP Italia	77,558	3.87	48, 72, 96	Floating Coupon	FOI
Foreign Debt	44,567	2.22	36, ..., 600	Fixed / Floating	Euribor, CPI

*Table 1 Italian Government debt: breakdown by instruments with their features.*

The key elements needed for a reliable and effective debt management process are the stochastic models for the evolution of interest rates and the mathematical representation of the cash flows as a function of the complete debt portfolio.

## Interest rate scenarios

A key input to implement the cost-risk analysis of the Italian public debt issuance portfolios are the out-of-sample scenarios for the term structure of interest rates relevant to the set of securities issued by the Italian Department of Treasury and to the set of derivative transactions. A class of models of the term structure of interest rates has been developed for the Italian sovereign bond term structure, the Italian break-even inflation (BEI) term structure, and for the swap curves of both Euro and US dollar. The main goal of these models is to produce joint stochastic out-of-sample scenarios for these different yield curves on the basis of calibrations that respect their observed in-sample properties, while ensuring internal consistency of the generated scenarios.

Starting from 2010, the development of the first release of the model was related to the sovereign (nominal) and break-even-inflation rates, with the purpose of pricing and forecasting nominal (BOT, CTZ and BTP) and inflation-linked (BTP€i) securities and then deriving endogenously the real curve. A similar approach has been adopted by the Federal Reserve for the estimation of the real curve referring to inflation-linked securities (TIPS). In this regard, see the work of Gürkaynak *et al.* (2008). Then the model was extended to the term structure of EUR swap rates in order to price and forecast the new sovereign (CCTeu) securities indexed to the Euribor rates and also allow the management of plain vanilla derivative contracts, such as interest rates swap with floating legs indexed to the Euribor rates. In 2012, the model was further extended including the specification of the term structure of USD swap rates, in order to manage cost-risk analysis on bonds denominated in foreign currency, price and forecast derivative contracts, such as cross currency swap, and improve the accuracy of EUR swap yield curve scenarios. The sovereign, break-even inflation (BEI) and EUR swap curves are estimated and simulated jointly, whereas the USD swap curve is modeled individually and so is considered exogenous to the others.

The last refinement of the model was then related to the development of a module linking the term structure of Italian break-even inflation rates to the scenarios generated for the Euro break-even rates, in order to price and forecast the securities designed for retail investors and indexed to Italian inflation, namely BTP Italia.

All these specifications are integrated in a module dedicated to the generation of stochastic scenarios, which interfaces the main module calculating cost and risk of the portfolios of issuances. The scenario generator also incorporates the possibility of using different stochastic models for generating medium/long-term scenarios for interest rates and inflation rates, allowing the evaluation of the expected performance, in terms of cost-risk analysis, of different strategies related to public debt issuance policies.

## Modelling the dynamics of interest rates

The approach adopted at the Department of Treasury for modelling the different term structures of interest rates requires the execution of multiple steps. The first step relates to the seminal work of Nelson and Siegel (1987), extended by Svensson (1994). The Nelson-Siegel-Svensson (NSS) approach uses an exponential components framework to characterize the entire yield curve within a four-dimensional parameters framework. This setup essentially decomposes the yield curve into four factors: the level, the slope and, two curvature factors of the yield curve.

A single yield curve can be represented as:

$$y(\tau) = \beta_1 + \beta_2 \left[ \frac{1 - \exp\left(-\frac{\tau}{\lambda_1}\right)}{\frac{\tau}{\lambda_1}} \right] + \beta_3 \left[ \frac{1 - \exp\left(-\frac{\tau}{\lambda_1}\right)}{\frac{\tau}{\lambda_1}} - \exp\left(-\frac{\tau}{\lambda_1}\right) \right] \\ + \beta_4 \left[ \frac{1 - \exp\left(-\frac{\tau}{\lambda_2}\right)}{\frac{\tau}{\lambda_2}} - \exp\left(-\frac{\tau}{\lambda_2}\right) \right] \quad (1)$$

where  $y(\tau)$  is the zero-coupon rate for maturity  $\tau$  and  $\beta_1, \beta_2, \beta_3, \beta_4, \lambda_1$  e  $\lambda_2$  are estimated parameters, which have an economic interpretation that implies the following restrictions:

$$\beta_1 > 0, \quad (\beta_1 + \beta_3) > 0, \quad \lambda_1 > 0, \quad \lambda_2 > 0.$$

This class of specifications is broadly adopted by many central banks and many financial institutions for the estimation of the term structure of interest rates.

The second curvature factor,  $\beta_4$ , allows more flexibility and is useful in the presence of multiple local optima and to better fit term structure data for long term maturities ( $> 10$  years), which is very relevant for the case of the Italian sovereign bonds. Indeed, the Italian Treasury issues bonds up to 50-year maturity and the allowance for a second curvature factor has been found to be crucial to capture the very long end of the term structure in our NSS estimation.

The NSS specification has two humps, and their positions are identified by  $\lambda_1$  and  $\lambda_2$ . These parameters can be either estimated or fixed. Setting fixed values for  $\lambda_1$  and  $\lambda_2$  yields lower volatility in estimation and better control of the evolution process of  $\beta$  parameters.

The main steps carried out in the development of the entire set of models are the following:

- unrestricted *in-sample* estimation, using daily data, of the NSS parameters for each curve: sovereign (Bot, Ctz, Cct, Btp), BEI (Zero Coupon Inflation Swap based on euro area HICP consumer price index), Euroswap (Euribor + Eurirs rates) and USD swap (Libor + USD swap rates);
- identification of average values of  $\lambda_1$  and  $\lambda_2$  for all curves and new model estimation with the calibrated values of  $\lambda_1$  and  $\lambda_2$ ;
- time aggregation from daily to monthly frequency (applying NSS fit to the end of month estimates);
- calibration of parameters and specification of the dynamics for all  $\beta_1$ - $\beta_4$  of all the term structures through a vector autoregression (VAR) approach for the joint evolution of the betas of the curves;
- *out-of-sample* stochastic simulation of the model.

The *in-sample* estimation of the yield curves allows to derive endogenously the real rates curve and the swap spreads (with respect to sovereign rates). In detail, if we define the break-even inflation (BEI) curve as the rates vector that makes equivalent (for investors) to hold nominal or inflation-linked securities, we get:

$$BEI_t(n) = y_t^{nom}(n) - y_t^{real}(n) \quad (2)$$

which implies

$$y_t^{real}(n) = y_t^{nom}(n) - BEI_t(n) \quad (3)$$

A similar approach is commonly adopted on US data for the estimation of the real curve referred to US inflation-linked securities (TIPS). See Gürkaynak *et al.* (2008).

Similarly, the swap spread curve is obtained endogenously as the difference between sovereign (nominal) and swap rates:

$$SPREAD_t(n) = y_t^{nom}(n) - y_t^{swap}(n) \quad (4)$$

for  $n = 0.25, 0.5, \dots, 30$  years.

The fit of the in-sample estimates is, in general, excellent and we observe very small pricing errors for all four curves, as showed in the following graphs (see Figure 1, Figure 2 and Figure 3).



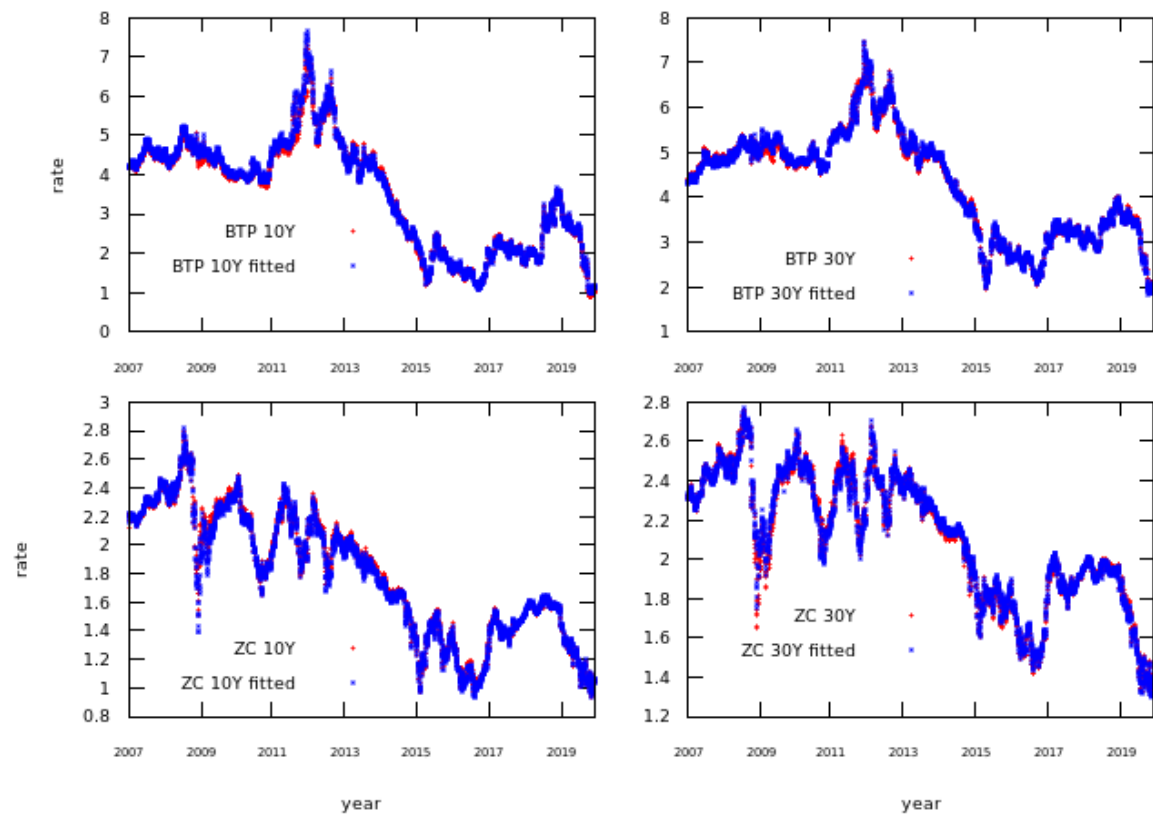


Figure 1 Nelson-Siegel-Svensson daily estimation (sovereign and break-even-inflation rates)

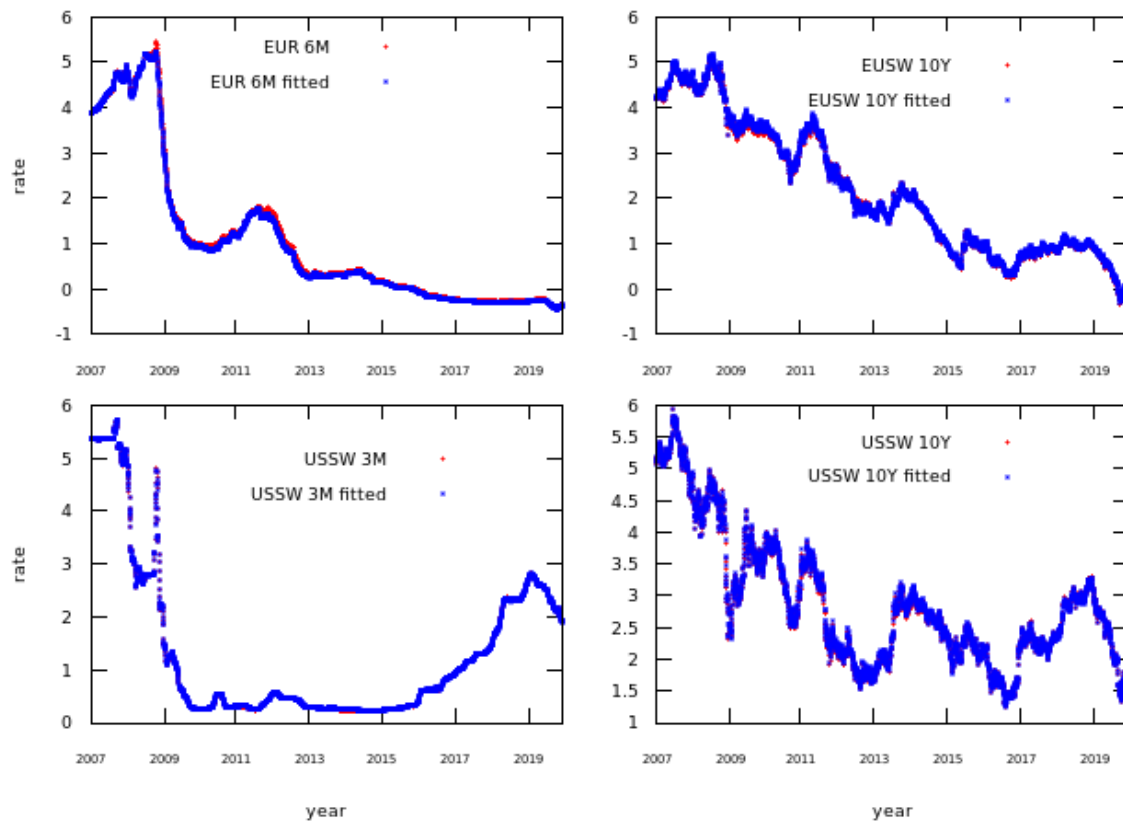


Figure 2 Nelson-Siegel-Svensson daily estimation (Euro and USD swap rates)



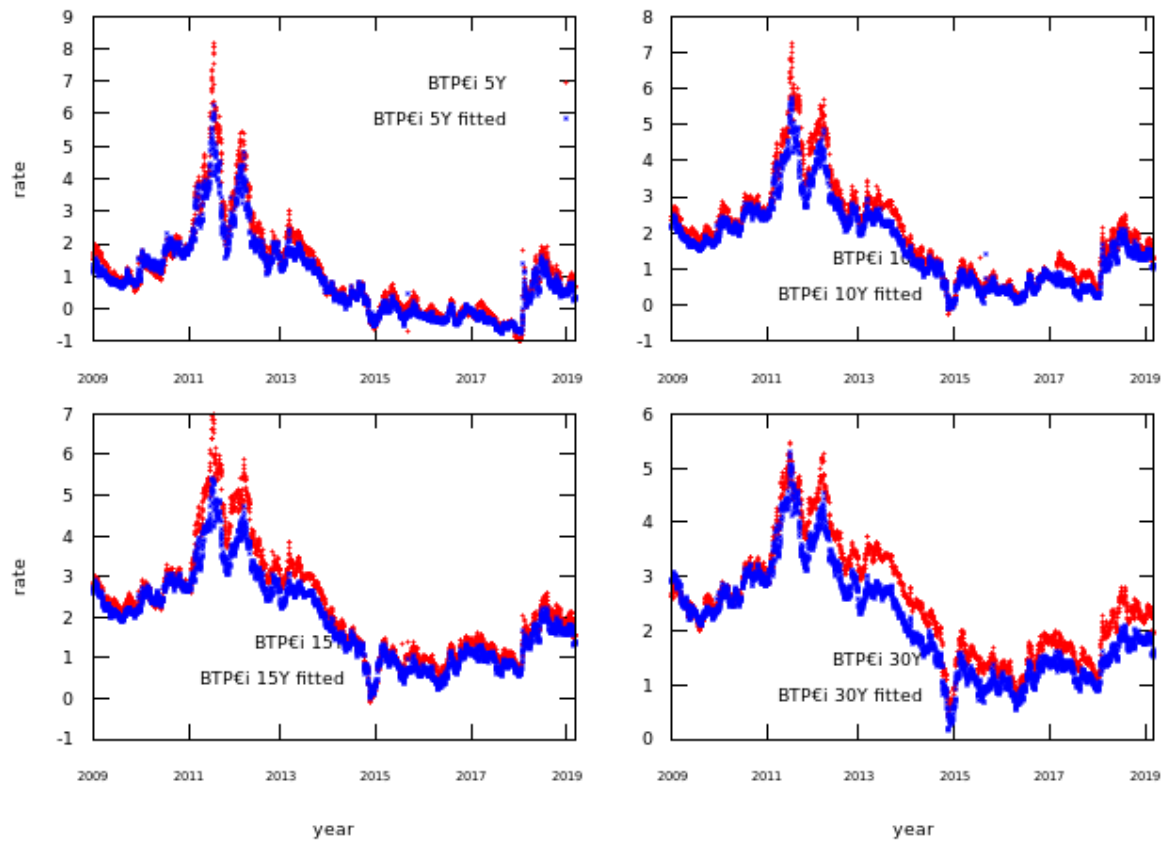


Figure 3 Nelson-Siegel-Svensson daily estimation (BTP€i real yields)

After in-sample estimation we perform a joint calibration of nominal, Euro BEI and euros swap curves that aims at finding out an evolution process for parameters  $\beta_1$ - $\beta_4$  that may produce out-of-sample forecasts (scenarios). As in Diebold and Li (2006), we adopt a multivariate specification based on a VAR model:

$$B_t = m + AB_{t-1} + e_t \quad (5)$$

where  $B_t$  is a 12-element vector for the  $\beta$  parameters (four for each of the three curves),  $m$  is a vector of intercepts,  $A$  is a 12x12 coefficient matrix describing the autoregressive scheme of parameters. The innovations  $e_t$  are assumed to be normally distributed in the estimation, which is carried out by maximum likelihood.

The VAR model, after estimation and general-to-specific model reduction designed to eliminate redundant coefficients<sup>1</sup>, is simulated out-of-sample to produce joint forecasts for the three sets of  $\beta$ s. This method ensures consistency for the out-of-sample scenarios of the term structures of sovereign (nominal), BEI and EUR swap rates. The USD swap curve is estimated and calibrated separately, and its level  $\beta_1$  enters in the level equation of the swap curve in euro.

<sup>1</sup> The general-to-specific procedure is performed iteratively, eliminating in every iteration the least significant coefficient in terms of its p-value.

The original multivariate specification ensured fulfillment of the statistical properties of yields, observed in-sample. These properties relate to no arbitrage conditions through the variance and covariance of the rates from the term structure models.

However, after the global financial crisis, some of the no-arbitrage conditions are inconsistent with the data. Therefore, some of the recent simulations allow for this feature in the scenario generation. Stochastic *out-of-sample* simulations of the model are performed using a *bootstrap*<sup>2</sup> method applied to the VAR residuals. At least 1000 different scenarios have to be *bootstrapped* in order to have stable results on average. The procedure is used to generate not only point forecasts, but also multiple sets of scenarios (distributions) consistent with the *in-sample* statistical properties of the yield curves. The parameters are regularly (re-)calibrated so as to include the most recent observations in the historical estimation sample and checking for structural stability, particularly after the shocks observed in recent years to the Italian sovereign rates and to the European swap curve.

Regarding the securities indexed to European and Italian inflation (BTP€i and BTP Italia) the scenarios produced for the BEI curve on Euro Area inflation are used to generate consistent projections for both the HICP (Euro Area) and FOI (Italy) consumer price indices.<sup>3</sup> In particular, the Italian break-even inflation rates are obtained through the simulation of a set of linear equations in which the Italian BEI rates are regressed on the Euro Area BEI rates. Further, considering the strong relationship between price indices and BEI behavior, we adopt a linear specification also for HICP and FOI, which are regressed on their lagged values and on the BEI values at 5-year maturity.

## Anchoring the yield curves using exogenous information on short term rates

When using the methods described above, it is often the case that there is a specific prior on the behavior of interest rates that can be used in calibration. The prior could be a specific belief of the issuer or the market expectation (survey) of one or more interest rates over a particular horizon. This is typically a prior about future short-term rates. In this case, it is desirable to "anchor" the relevant rates and generate simulations that condition on such prior. In recent years a new module for "anchoring" yield curves has been developed, starting from the work of Altavilla *et al.* (2013), who proposed a methodology for tilting (or rotating) the yield curve using exogenous priors on short term rates (e.g. survey data).

The anchoring technique exploits the informational advantage of survey expectations about short yields in order to improve the accuracy of yield curve forecasts given by a base model. The anchoring methodology is independent from the base model, and the basic idea is to tilt the yield curves

---

<sup>2</sup> The innovations generated during the stochastic simulation with the bootstrap method are extracted randomly (with repetition) from the set of residuals of the stochastic equations of the  $\beta$  parameters observed in-sample. The use of this technique, compared to the extraction from a normal distribution (Monte Carlo), may be more appropriate in cases where the residuals of the dynamic equations are not normally distributed.

<sup>3</sup> Treasury Bonds linked to Euro-zone inflation are securities that provide investors with protection against price increases. Both the principal to be redeemed at maturity and their coupons, paid semiannually, are adjusted for inflation in the Euro-zone, as measured by the Harmonised Index of Consumer Prices (HICP), excluding tobacco. Monthly data for the Eurostat index may be found at the Statistical Office of the European Communities website (Eurostat). Btp Italia is a government security that provide investors with the protection against an increase in the level of prices in Italy: both the coupons, which are paid semi-annually, and the principal, the revaluation of which is also paid semi-annually, are indexed to the Italian inflation, as measured by ISTAT – the Italian National Bureau of Statistics - through the FOI national index, "Prezzi al consumo per le famiglie di operai e impiegati", with the exclusion of tobacco products.

generated by the base model, incorporating the exogenous priors on short rates, without re-estimating the parameters or changing the original specification.

The method is based on the projection of the out-of-sample density forecasts  $f_t$  onto the space of densities that have conditional mean equal to the exogenous prior and that are closest to  $f_t$ , according to a Kullback-Leiber measure of divergence. Basically, if the vector of expected yields  $h$ -periods ahead has mean  $\mu_{t+h}$  and variance-covariance matrix  $\Sigma_{t+h}$ , and at time  $t$  we have survey data on the first (i.e., shortest)  $r$  maturities related to period  $t + h$ ,  $\mu'_{t+h,1:r}$ , the tilted (anchored) vector is calculated, starting from the average scenario of the model, as follows:

$$\mu_{t+h}^* = \left( \begin{array}{c} \mu'_{t+h,1:r} \\ \mu_{t+h,r+1:m} - \Sigma_{t+h,21}(\Sigma_{t+h,11})^{-1}(\mu_{t+h,1:r} - \mu'_{t+h,1:r}) \end{array} \right) \quad (6)$$

where the variance-covariance matrix  $\Sigma_{t+h}$ , can be rewritten as:

$$\Sigma_{t+h} = \left( \begin{array}{cc} \begin{array}{c} \Sigma_{t+h,11} \\ r \times r \end{array} & \begin{array}{c} \Sigma_{t+h,12} \\ r \times (m-r) \end{array} \\ \begin{array}{c} \Sigma_{t+h,21} \\ (m-r) \times r \end{array} & \begin{array}{c} \Sigma_{t+h,22} \\ (m-r) \times (m-r) \end{array} \end{array} \right)$$

If we consider, for example, the anchoring to an exogenous value of the rate with the shortest maturity ( $r = 1$ , e.g., the 3-months rate), the projection of the single term structure produced by the dynamics of the factors of the Nelson-Siegel-Svensson curve is obtained by setting the 3-months rate equal to the exogenous value (prior) and modifying the data for the other maturities according to equation (6).

Recently the anchoring module has been implemented into the scenario generator of SAPE and so this technique can be applied to any single scenario of the yield curves generated with the stochastic simulations of the baseline term structure model.

The first release of the anchoring module performed the tilt of the curves only on the first out-of-sample observation, in line with the results of Altavilla *et al.* (2013). Usually the exogenous prior is the sovereign rate of the zero-coupon instrument with the shortest maturity (BOT 3 months). Moreover, considering that the term structures of sovereign, BEI and EUR swap rates are simulated jointly, we developed a method for selecting the priors also on BEI and EUR swap rates, so as to ensure the consistency with the one set on the 3-months nominal rate. In detail, we set the other two priors taking the average of the 3-month euribor rates and of the 2-year BEI rates calculated on the first out-of-sample observation of the five baseline scenarios which are closest to the 3-months nominal rate prior, with the constraint that the selected scenarios must have the same direction (sign) in terms of rotation.

The second version of the anchoring module has been recently released, that makes possible the rotation (tilt) of the curves not only for the first out-of-sample observation, but also for the following ones, for example up to 12 months onwards. This extension allows to simulate the model, for example, granting the consistency with different scenario hypotheses related to the expectations in terms of issuance policy and monetary policy choices.

## The debt portfolio cash flow and the balance constraint

By debt portfolio we mean the collection of domestic securities issued by the Italian Treasury that are still on the market, that is securities that have not yet reached their maturity. A quantitative description of the debt portfolio benefits from establishing precise notation and the definitions:

- The index  $k$  represents the type of security, where  $k \in \{1..N_B\}$  and  $N_B$  is the number of different securities issued by the Italian Treasury. Securities of the same kind but with different maturities are considered different;
- The index  $t$  represents the months both in the balance period and in the planning period, where  $t \in \{T_s..T_f\}$ ,  $T_s$  is the starting month and  $T_f$  is the final month;
- $u_k(t)$  indicates the nominal value of securities of type  $k$  issued at time  $t$ ;
- $m_k(t)$  indicates the maturity, measured in years, of securities of type  $k$  issued at time  $t$ ;
- $p_k(t)$  indicates the price of securities of type  $k$  issued at time  $t$ ;
- $N_{c_k(t)}$  indicates the number of coupons that will be paid for bonds of type  $k$  issued at time  $t$ ;
- $c_k(t)$  indicates the fixed coupon percentage for bonds of type  $k$  issued at time  $t$  or the fixed real coupon percentage for inflation indexed bonds of type  $k$  issued at time  $t$ ;
- $c_k(l; t)$  indicates the floating coupon percentage paid for the coupon  $l$  of bonds of type  $k$  issued at time  $t$ ;
- $m_k(l; t)$  indicates the time to maturity, measured in years, of the coupon  $l$  of bonds of type  $k$  issued at time  $t$ ;
- $s_k(t)$  indicates the spread for the coupons of bonds  $k$  issued at time  $t$ ;
- $r_d(T; t)$  indicates the zero coupon rate for domestic issuances for maturity  $T$  observed at time  $t$ ;
- $r_s(T; t)$  indicates the euro swap rate for maturity  $T$  observed at time  $t$ ;
- $HICP(t)$  and  $FOI(t)$  are the European and Italian Inflation Indexes, respectively;
- $I_k(T; t)$  indicates the index coefficient at time  $T$  of bonds of type  $k$  issued at time  $t$ .

To describe the cash flows for the portfolio, we use the indicator function  $\theta(s; t)$ , whose value is 1 if  $s = t$ , 0 otherwise. The unitary cash flow at month  $s$  for bond  $k$  issued at time  $t$  is composed by three terms:

- The net income at issuance time  $t$ :

$$N_k(s; t) = \theta(s; t) \frac{p_k(t)}{100};$$

- The coupon payment at time  $m_k(l; t)$  with  $l \in \{1..N_{c_k(t)}\}$

$$C_k(s; t) = \sum_{l=1}^{N_{c_k(t)}} \theta(s; t + m_k(l; t)) I_k(m_k(l; t); t) c_k(l; t);$$

- The reimbursement at time  $m_k$

$$R_k(s; t) = \theta(s; t + m_k) I_k(m_k; t).$$

Using the above defined functions, it is possible to represent the cash flow for the whole portfolio of domestic bonds at any time  $s$  as follows:

$$F_d(s) = \sum_{k=1}^{N_B} \sum_{t=s-m_k}^s u_k(t) (N_k(s; t) - C_k(s; t) - R_k(s; t)).$$

Besides the cash flow of the domestic portfolio, it is necessary to include the Primary Budget Surplus (PBS), the cash flow of the securities issued in foreign currency and derivative transactions in the balance equation. As to the Primary Budget Surplus, any forecast is difficult due to issues like political decisions and changes in the state of the economy. However, we assume that the PBS is given as an input parameter every month so defining a sequence  $PBS(s)$ . As to the cash flow in foreign bonds and derivatives, we consider it as another given quantity  $F_e(s)$ ; however that quantity may be adapted according to the simulated interest rates scenarios for those components in  $F_e(s)$  which are indexed to inflation or to some interest rate (e.g., an interest rate swap). The cash flow of bonds' issuances and payments goes through a Bank of Italy account, owned by the Treasury, called Treasury Cash Account. By regulation, there is a positive lower bound,  $\beta = 10$  billion euro, on the amount of money that account must hold. We indicate by  $TCA(s)$  the amount of money in the Treasury Cash Account at time  $s$ . Then the balance constraint, i.e. the constraint that must be respected in order to balance the income and expenditure of the public debt, can be written as

$$TCA(s) = TCA(s+1) + F_d(s) + F_e(s) + PBS(s) \geq \beta.$$

This is the fundamental constraint that guarantees the payment of coupons and the reimbursement of bonds at maturity. For the historical issuances, the constraint has necessarily been respected, whereas in the simulation time window the issued quantities  $u_k(t)$  have to be adapted to fulfill the balance constraint.

## The ESA 2010 cost function

The definition of a suitable objective function is an element of pivotal importance in any decision making process. In public debt management, there is a natural cost to consider that is the cost of the debt service, i.e. the interest expenses. Actually, there are a number of possible choices for quantifying the interest expenses. However, since it falls within the regulatory parameters of the European community, we follow the European System of National and Regional Accounts (ESA 2010), that is the latest internationally compatible EU accounting framework for a systematic and detailed description of an economy.

The ESA 2010 criteria refer to a given time period, say  $[t_1, t_2]$ , and, loosely speaking, consider for each bond its total cost (original issue discount, coupons and uplift for securities inflation indexed) as distributed over its existence period, namely from issuance to maturity, taking the portion of the cost that has an intersection with the considered time period of the ESA 2010. To be more precise, each cost associated to a bond has a natural reference period (e.g., six months for a coupon or the complete lifetime of the bond for the original issue discount) that can be indicated as  $[t_s, t_e]$ , and impacts on the ESA 2010 proportionally to the ratio between (N), the number of days of intersection between the ESA 2010 period and the reference period of the cost, and (D), the number of days of the reference period of the cost. In the following, we present the impact on the ESA 2010 of the various types of costs for a unitary quantity of a security of type  $k$  issued at time  $t$ :

- ESA 2010 cost in the period  $[t_1, t_2]$  for the original issue discount

$$E_k^{oid}([t_1, t_2]; t) = \left(1 - \frac{p_k(t)}{100}\right) \frac{[t_1, t_2] \cap [t, t + m_k(t)]}{[t, t + m_k(t)]};$$

- ESA 2010 cost in the period  $[t_1, t_2]$  for the coupons of the security of type  $k$  issued at time  $t$

$$E_k^{coup}([t_1, t_2]; t) = \sum_{l=1}^{N_{c_k(t)}} I_k(m_k(l; t); t) c_k(l; t) \frac{[t_1, t_2] \cap [t + m_k(l-1; t), t + m_k(l; t)]}{[t, t + m_k(t)]};$$

- ESA 2010 cost in the period  $[t_1, t_2]$  for the uplift

$$E_k^{uplift}([t_1, t_2]; t) = I_k(\min[t + m_k(t), t_2]; t) - I_k(\max[t, t_1]; t).$$

Finally, the ESA 2010 cost in the period  $[t_1, t_2]$  for the whole portfolio of domestic bonds can be represented as

$$E([t_1, t_2]) = \sum_{k=1}^{N_B} \sum_{t=t_1-m_k}^{t_2-1} u_k(t) \left( E_k^{oid}([t_1, t_2]; t) + E_k^{coup}([t_1, t_2]; t) + E_k^{uplift}([t_1, t_2]; t) \right).$$

### Analytical study on the ESA 2010 evolution

The above equation is a very general representation of the interest expenditure, since it accounts for any kind of security. Hereafter, we limit our investigation of the ESA 2010 dynamics to two types of securities, zero-coupon bonds and bonds with fixed coupons, and assume that:

1. each security with coupons is issued at par value, i.e.,  $p_k(t)=100$ ;
2. each security is issued the first day of a month and the time frame for the evaluation of the ESA 2010 cost is always an integer number of months;
3. the amount of each security issued every month does not depend on time but only on the type of security. If  $D$  is the initial debt at time  $t = 0$ , we suppose that  $D$  is distributed, according to a fraction  $x_k$ , among the different securities  $k$  and the amount for each security is uniformly distributed across the maturity,  $m_k$ , of the security so that:

$$u_k = \frac{D x_k}{m_k}.$$

The third assumption is the most important one. It means that the total debt and its composition remain constant: only maturing securities require new issuances. Coupons are paid, for instance, using the primary budget surplus. A simplified version of the ESA 2010 interest expenditure is useful to start the analysis of the monthly interest expenditure. The assumption that the government issues either zero-coupon bonds or par value bonds makes it possible to derive the ESA 2010 interest expenditure as a function of past interest rates. In particular, the ESA 2010 requirement of spreading the original issue cost along the whole life of a security implies that the cost of a zero-coupon bond in a given month,  $M$ , during the life of the bond  $k$  issued at time  $t$ , is equal to

$$E_k^{oid}(M; t) = \frac{r_d(m_k(t); t)}{12}$$

and a similar relation holds for the ESA 2010 costs of a coupon

$$E_k^{coup}(M; t) = \frac{r_d(m_k(t); t)}{12}$$

since, for fixed coupon bonds issued at par value,  $c_k(t) = r_d(m_k(t); t)/2$ , and that cost has to be distributed along six months. We can now define  $\varphi(T)$  as the total ESA 2010 cost in a single month,  $T$ , as

$$\varphi(M) = \sum_{k=1}^{N_B} \sum_{t=M-m_k}^{M-1} u_k(t) \frac{r_d(m_k(t); t)}{12} = \sum_{k=1}^{N_B} \sum_{m=1}^{m_k} u_k(M-m) \frac{r_k(M-m)}{12},$$

where, to simplify the notation, we use the change of variables  $m = M - t$  and  $r_k(t) = r_d(m_k(t); t)$  without ambiguity, since in this paragraph we limit our analysis to a fixed domestic bond;  $r_k(t)$  indicates the spot rate for issuances of the domestic bond  $k$  having maturity  $m_k$  issued at time  $t$ . Assumption 3 simplifies the above expression since  $u_k(t)$  is fixed, depending only on the portfolio composition

$$\varphi(M) = \frac{D}{12} \sum_{k=1}^{N_B} \frac{x_k}{m_k} \sum_{m=1}^{m_k} r_k(M-m).$$

If we define the average value of rate  $r_k(t)$  in the period between  $[M - m_k, M - 1]$  as

$$R_k(M) = \frac{1}{m_k} \sum_{m=1}^{m_k} r_k(M-m)$$

the monthly ESA 2010 becomes

$$\varphi(M) = \frac{D}{12} \sum_{k=1}^{N_B} x_k R_k(M).$$

We are now ready to write down the ESA 2010 simplified expression:

$$E([t_1, t_2]) = \sum_{M=t_1}^{t_2} \varphi(M) = \frac{D}{12} \sum_{k=1}^{N_B} x_k \sum_{M=t_1}^{t_2} R_k(M) = \frac{D}{12} \left( \sum_{k=1}^{N_B} x_k R_k \right) (t_2 - t_1)$$

where  $R_k$  is the average of the spot rate associated to the security  $k$  in the period  $[t_1, t_2]$ . The above expression shows that, using reasonable approximations, the ESA 2010 cost has a very intuitive expression as a linear function of time with a coefficient that is the composition of the portfolio percentages multiplied by the average rates. The actual ESA 2010 dynamics that we compute with our software tool shows that the linear approximation can be considered a good proxy and the growth rate of the ESA 2010, i.e., the function  $\varphi(M)$ , can be considered as a simple but informative cost function.

## The Cost/Risk Analysis

Both mathematical models, the one used for the generation of possible scenarios of future interest rates and the other describing the cash flows of the public debt and the ESA 2010 cost function, can be used separately and in combination in several ways. First of all, it is possible to compute financial indicators useful for the compilation of accurate balance reports, since the model considers the real outstanding debt (including also issuances in foreign currencies and derivative contracts). It is also



possible to estimate future interest expenditures due to changes either in the composition of the portfolio of public debt securities or to the dynamics of interest rates, *e.g.*, if they remain low for the next five years or they undergo shocks (what-if analysis). However, the most interesting analysis is probably the one based on Monte Carlo simulations in which the performance of different possible funding portfolios is compared with respect to a large number of possible scenarios of interest rates evolution. This is the so called *cost-risk* analysis that is a fundamental task that the public debt manager must be able to perform in a very reliable way. In the previous subsection, we described, in analytic form, the dynamics of interest-expenditure for a portfolio under certain assumptions. Hereafter, we extend and deepen the analysis considering the actual evolution of interest expenditure without imposing any constraint and we determine the probability distribution of the ESA2010 interest expenditure with respect to a stochastic distribution of interest-rate paths.

### Classical measures of cost and risk

Classically, the choice of the debt portfolio relies on a trade-off between cost and risk measures that must be statistically defined and based on the government objective function. The most commonly used cost and risk measures are the mean and the standard deviation of the objective function (the ESA 2010 in our case), respectively, but other functions have been developed (see Fig. 4 for a pictorial representation of cost and risk measures) which are more suitable for the management of public debt, like the Cost at Risk.

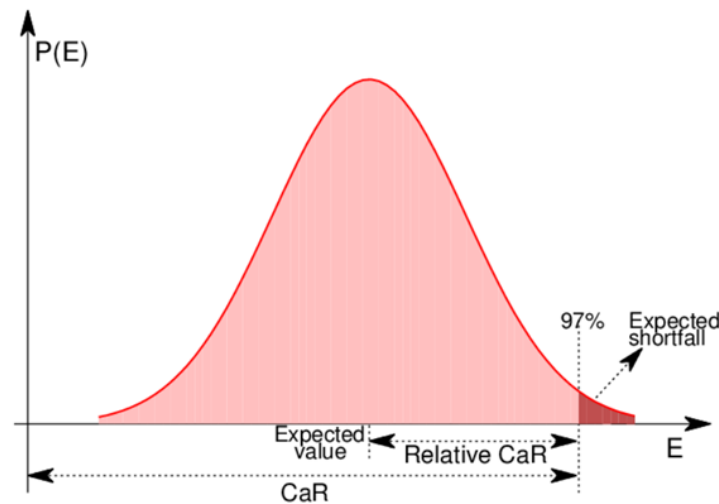


Figure 4. Schematic representation of the ESA2010 Probability Distribution and associated cost and risk measures.

The Cost at Risk (CaR) is a key reference indicator in debt strategies simulation analysis because it depends on both the expected value and risk characteristics of the probability density function of the interest expenditure (Jorion 2006). The CaR is the maximum value of the interest expenditure that can be expected with a probability  $p$  over a given time horizon. In formula

$$\text{CaR}_p = \min_x \{x \mid \text{Prob}(E([t_1, t_2]; P, S) < x) \geq p\},$$

where we extend the notation introduced in the *ESA 2010 cost function* Subsection indicating with  $E([t_1, t_2]; P, S)$  the ESA2010 cost in the period  $[t_1, t_2]$  for a portfolio  $P$  of public debt securities, and for a scenario  $S$  of the interest rates' evolution. The quantity  $E([t_1, t_2]; P, S)$  can be considered a stochastic variable since, for any portfolio  $P$  it depends on the scenario  $S$ . The CaR indicates the interest expenditure that the Government can expect not to exceed with a confidence level  $p$ . It offers a simple metrics of the interest cost that the government could incur under unfavorable circumstances. The higher the probability  $p$ , the greater is the consideration of unfavorable realizations of the interest

expenditure and thus the greater is the attention to risk in the choice of the debt portfolio. Two other metrics of risk are often considered in relation to the CaR: the Relative Cost at Risk and the Expected Shortfall. The Relative CaR is given by the difference between the CaR and the expected value of the interest expenditure over a given horizon. It can be seen as a measure of the width of the interest expenditure probability density function. It is defined as

$$\text{RelCaR}_p = \text{CaR}_p - \langle E([t_1, t_2]; P, S) \rangle_S,$$

where, hereafter, with  $\langle (\cdot) \rangle_S$  we indicate the average value of the quantity  $(\cdot)$  with respect to the random variable  $S$ .

The Expected Shortfall (ES) is the expected value of the interest expenditure in case the CaR measure is exceeded. It is defined as

$$\text{ES}_p = \langle E([t_1, t_2]; P, S) \rangle_{S|E \geq \text{CaR}_p}.$$

ES indicates the expected interest expenditure conditional on the realization of the  $1 - p$  percentage of worst possible scenarios. It provides an estimate of the expected interest expenditure under extremely unfavorable circumstances. As the ES depends on the thickness of the upper tail of the probability distribution of interest expenditure, it complements the CaR measure in case of non-normality of the distribution of interest expenditure. As a matter of fact, it is worth noting that if the distribution of interest expenditure were normal, the above-mentioned statistical indicators would be fully determined by the mean and the standard deviation of the distribution.

## Evolution measures of cost and risk

Before showing an example of cost-risk analysis, it is worth noting there is an important limit in the use of the statistical measures just described: they refer to a fixed time interval. The consequence is that the value of all the measures, and, therefore, the cost/risk analysis, depends on the chosen time interval. However, by using the results presented in the *Analytical study on the ESA 2010 evolution* subsection, in which we showed that, under certain assumptions, there is a linear dependence of the ESA2010 cost on the time interval, we may introduce new quantities that provide robust measures of cost and risk that are only loosely dependent on the time window. As a first example, in the left panel of Figure 5 we show the ESA2010 growth dynamics of the actual portfolio of the Italian government bonds on December 12, 2019 according to two hundred possible scenarios of interest rates.

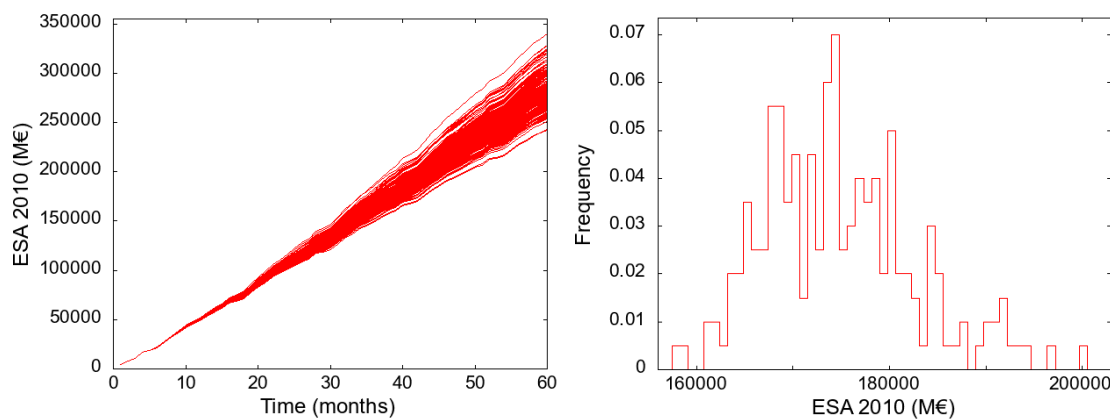


Figure 5. ESA2010 growth path (on the left) and ESA 2010 distribution after three years of simulation (on the right).

At any time of the simulation period (5 years in the present case), it is possible to compute the distribution of the ESA2010 interest expenditure (right panel). As a matter of fact, as showed in Figure

5 also the growth of the ESA2010 cost computed exactly, without any approximation, can be considered, with a good degree of approximation, as linear. In a way similar to what presented in the *Analytical study on the ESA 2010 evolution* subsection, we indicate with  $\varphi(M; P, S)$  the ESA2010 cost in month  $M$  for a portfolio  $P$  of public debt securities and for a scenario  $S$  of the interest rates' evolution. To obtain the ESA2010 cost in the time window  $[t_1, t_2]$  it is sufficient to compute the sum of the function  $\varphi(M; P, S)$  in the period  $[t_1, t_2]$ :  $E([t_1, t_2]; P, S) = \sum_{M=t_1}^{t_2} \varphi(M; P, S)$ . Therefore, for the purpose of computing a reasonable approximation of the growth rate of the ESA2010 cost, it is possible to use the function  $\varphi(M; P, S)$ . However, since the actual growth of the ESA2010 is not perfectly linear, different growth rate metrics may be defined for specific needs of the public debt manager. It is possible to consider the initial growth rate  $R_{t_1}(P, S) = \varphi(t_1; P, S)$  (if one is interested in the early evolution of the debt), the best growth rate  $R_F(P, S) = \frac{\sum_{M=t_1}^{t_2} (M-t_1) \cdot E([t_1, M]; P, S)}{\sum_{M=t_1}^{t_2} (M-t_1)^2}$  (if one is interested in the typical growth), or the final growth rate  $R_{t_2}(P, S) = \varphi(t_2; P, S)$  (if one is interested in the long term growth).

Since with Monte Carlo simulations it is possible to evaluate the performance of the portfolio  $P$  over a large number of scenarios, we may define for each of the above mentioned growth rates a probability distribution and the corresponding measures of costs and risks. The idea is that instead of computing the costs and risks measures (such as the CaR) over the ESA2010 distribution at a fixed time, they are computed on the distribution of the chosen growth rate. Therefore, assuming, for instance, that a debt manager is interested in the typical growth rate of the ESA2010 cost, it is possible to generate an efficient frontier, associated to different debt portfolios, by using the average of the best fit to the growth rate  $\langle R_F(P, S) \rangle_S$  and its standard deviation  $\sigma(R_F(P, S))$  with respect to a large enough set of scenarios (see Figure 6).

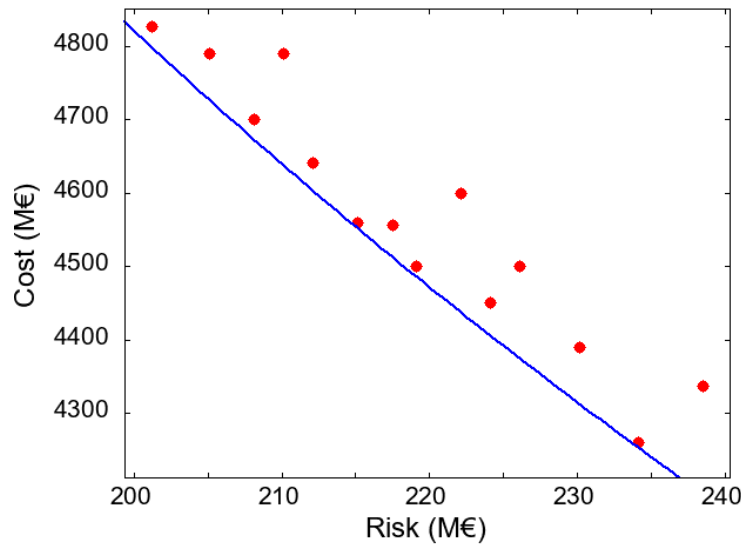


Figure 6. Efficient Frontier.

## CVA modelling

This Section describes the development of the mathematical model and the scenarios generator, which are implemented in a satellite module of the system dedicated to the calculation of the Credit Valuation Adjustment (CVA) for collateralized derivative transactions.

Prior to introducing the details of the CVA, it is important to understand its purpose within the context of the so called Credit Support Annex (CSA). A CSA is an agreement that regulates the credit support (collateral) to mitigate the credit risk of OTC derivative transactions. It defines the conditions under which, at given dates, collateral is posted, or transferred, between the counterparties that enter into an OTC derivative contract, according to the following elements:

- *frequency*: time interval between two contiguous collateral postings;
- *minimum transfer amount*: the smallest amount of collateral exchanged; below this amount no collateral is posted;
- *threshold*: reference mark-to-market above which the collateral should be posted.

In order to meet specific needs, Italian Treasury customized the canonical CSA framework adding a potentially fourth element to limit the maximum amount of collateral to be posted at each date. This leads to a non-standard framework with increasing cash flows, which gradually should ensure a coverage of the credit risk with a limited burden for Italian Treasury.

Posting collateral against a derivative position usually cancels the credit risk of that position. Compared to a standard CSA, a non-standard framework reduces the credit risk of a certain amount. The metrics to measure this amount is then given by the Credit Valuation Adjustment.

Given a liability portfolio, the CVA is the difference between its value in absence of risk and the value calculated considering a counterparty's default. In other words, it is the credit adjustment applied to derivatives transaction to account for counterparty's default.

Let  $L_{spot}$  be the expected losses that can occur in the time interval  $[0, T]$ :

$$L_{spot} = lgd \cdot EE \cdot DP$$

where  $lgd$  is the loss given default,  $EE$  is the expected exposure and  $DP$  is the default probability. Assuming the recovery rate is equal to  $(1 - lgd)$  and constant at default time, the CVA may be computed with the following expression:

$$CVA = lgd \cdot \int_0^T E^Q[D(t) \cdot EE \mid t = \tau] dDP(0, t)$$

The CVA calculation begins with the generation of a set of  $n$  scenarios. A scenario is a set of  $m$  forward-looking, monthly term structures of interest rates, each one representing the spot curve at  $i$ -th month in the future. The number of months usually fits the time-to-maturity of the longest swap in portfolio.

The scenarios are generated through the specification of a standard one-factor Hull & White model. With this model, an array of instantaneous forward rates (short rates), one per each month of simulation, is derived from the following formula:

$$dr(t) = [\theta(t) - \alpha r(t)]dt + \sigma dW(t)$$

where:

- $r(t)$  is the short rate
- $dW(t)$  is a Wiener process  $\sigma$  is the volatility of the short rate
- $\alpha$  is the mean reversion rate
- $\theta$  is a drift function

From the simulation of the short rate's path, a full yield curve is generated at each simulation date through an affine function. This term structure represents the single scenario and is composed by one hundred maturities, where each point is the forward rate calculated every six-months, from the maturity of six months up to fifty years.

Latest finance guidelines suggest discounting the future value of a derivative using the factors from an Overnight Indexed Swap curve. This requires to generate two correlated sets of scenarios having the same cardinality: the first on the swap IBOR curve, to determine the forward rates, the second on the swap OIS curve, to compute the discount factors.

The calculation of mark-to-markets is iterated for every swap in portfolio, for each of the  $n$  generated scenarios, devising an  $n \times m$  array of mark-to-markets, where  $m$  is the number of months between the simulation date and the expiry of the swap with the longest maturity.

Generating the raw mark-to-markets rather than the final  $EE$  allows to perform further analysis and post-processing, such as the application of the non-standard CSA framework. In this case it has been adopted a two-step procedure where in the first step every mark-to-market is recalculated according to the non-standard CSA constraints described above, while in the second the positive filtered mark-to-markets are separated from the negative ones. Finally, both subsets of mark-to-markets are grouped into two  $m$ -sized arrays, which actually contain the Expected Positive (EPE) and Negative (ENE) Exposures for each month of simulation, as shown in figure 7.

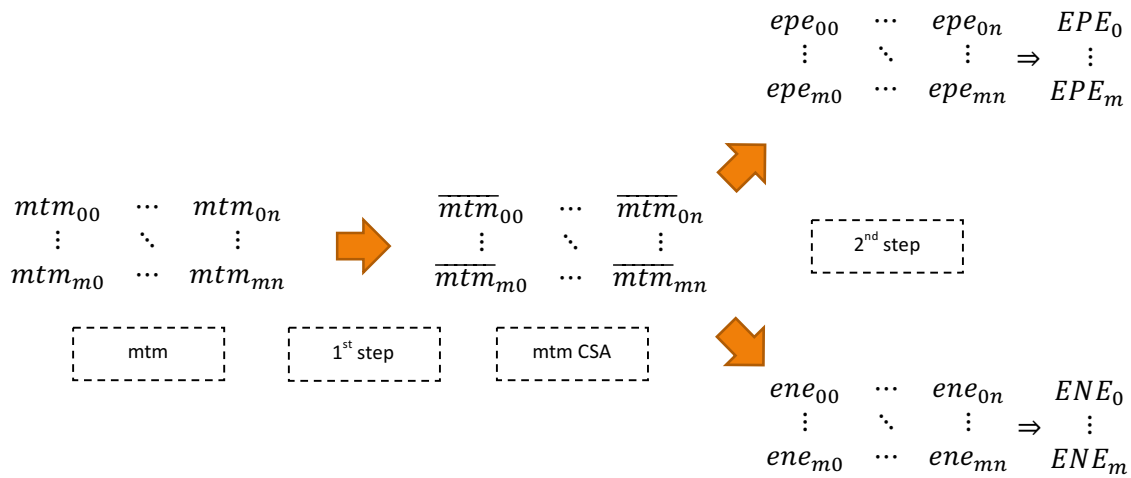


Figure 7. EPE and ENE calculation.

Both sets of mark-to-market are stored in a  $m$ -sized array.

For each simulation month the Expected Exposure value is discounted by the spot OIS curve and multiplied by the corresponding default probability (DP) derived from the term structure of Credit Default Swap using a standard *moral hazard* model. The final value of CVA comes from the sum of all values in the dataset multiplied by the Loss Given Default.

Since the computational cost of this procedure is incompatible with response time requested by counterparties, there is the need to optimize it by processing multiple scenarios at once. Thus this application follows a multi-threaded approach which ensures a speedup that grows linearly with the number of threads.

In order to manage a heterogeneous set of instruments the logic behind calculations has been moved into a pricing library, to which the application is linked. Currently this library has the ability to price the following types of instruments:

- interest rate swaps
- cross currency swaps
- swaptions

Each instrument is modelled through a class which encloses complex data structures and algorithms needed to perform the calculation of mark-to-market. The major advantages of this solution are:

- reusability: the pricing library can be used, and effectively it is, by other applications
- extensibility: because of its structure, functionalities can be easily added to the library

For example, over time the library has been enriched to manage some features of the swap, such as amortizing swaps, different payment frequencies between the two legs of a swap and break clauses. The last functionality models the possibility that a swap may be terminated before its natural expiration (early termination option (ETO)), leading to a second dataset of raw mark-to-markets. In this dataset, the swap's cashflows after the early termination date are simply discarded and thus the contribution of the swap to the mark-to-market of the portfolio after that date is null.

## Software Architecture

The Sistema Analisi Portafogli di Emissione (SAPE) software is a quite sophisticated tool [1], made of a set of modules that interact each other through specific Application Programming Interfaces (API) and conventions.

The core of the system is the *Computational Engine (CE)*. The CE determines debt cash flows considering the outstanding government bonds and the future (simulated) interest rates. The CE indicates what will be the future issuances of any kind of security based on a given portfolio composition and an exogenous estimate of the budget surplus. In the CE there is full integration among the domestic debt, the foreign currency debt and the derivatives instruments, that results in a complete and detailed description of the portfolio of government securities. The CE computes all the quantities of interest for the public debt manager (*e.g.*, the debt cost according to the ESA2010 criteria, the *duration* of the portfolio, *etc...*) and provides input to the Cost/Risk analysis module. The CE can be used also as an accounting tool to double-check the output of other commercial software products in use at the Italian Treasury for the management of public debt. The CE module is written in C language for efficiency reasons.

A crucial component of the SAPE is the *Interest Rates generation* module that simulates the evolution of interest rates according to the different stochastic models described earlier in this paper, including, among others, the VAR model and the Hull-White one-factor model. All the scenario generators are written in C language.

The *Cost/Risk* module analyses and shows the results of Monte Carlo simulations comparing the performance of different portfolios under several measures of cost, including both the ESA2010 accrual cost and the cash-flow costs in a given time interval. The *Cost/Risk* module makes possible to trace an efficient frontier of the public debt, by choosing suitable portfolios and cost and risk measures. That module is developed part in C language and part in the Perl scripting language. One of the reasons

for using Perl is that at the time we started the development of the SAPE, Perl offered a better support for managing (*i.e.*, reading and writing) Excel sheets, that are one of the main output formats required by the Italian Treasury staff.

Finally, there is the CVA module, designed to analyze collateralized derivative transactions as described in the previous Section.

The SAPE software requires input data coming from multiple sources:

- i) the *GEDI* (GEstione Debito Italiano) system that provides all the information about new issuances, derivatives, non-domestic securities and interest rates real-time and historical data. Those data are stored in a relational database and retrieved using a Web service;
- ii) the *SAPEDB* database of historical issuances that provides information about the outstanding domestic bonds and the historical values of some macro indexes (euribor, HICP, FOI);
- iii) the Interest rates generation module.

Finally, there is an output module that stores all the relevant files of each simulation in a SQLite database. The database makes search and retrieval operations very easy and efficient. This, in turn, facilitates, for instance, the set-up of simulations that are like others carried out in the past. In addition, this module also deals with the creation of graphs and reports to present the results of numerical simulations in a way that is easy to interpret.

SAPE offers a comprehensive interactive GUI based on Perl/Tk that includes a context-sensitive help. However, it is possible to run many of the software modules also in background to produce results in *batch* mode. In particular, the interest rates generation module can produce thousands of scenarios that can be used for other purposes or simply to carry out a detailed statistical analysis of their behavior. All the software components are highly portable, and they may run on Mac OS X, Linux and Windows, the latter being the platform in use at Italian Treasury premises.

Figure 8 summarizes the SAPE organization and the relations among the modules. In particular, SAPE DB, GEDI and the Interest rate generator are data sources, whereas the CE and the CVA carry out the required computations whose results are analyzed and saved in the remaining two modules.

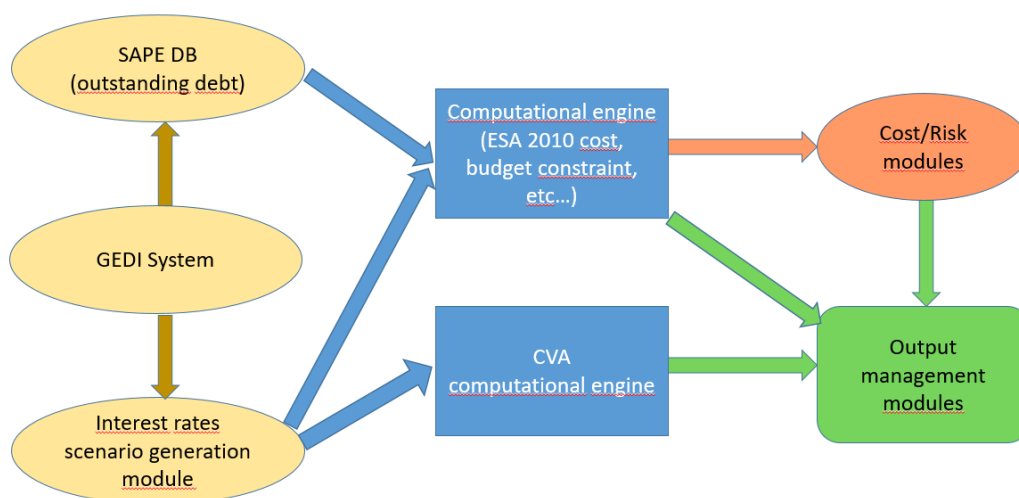


Figure 8 The organization of the SAPE software



## Conclusions

The models used by the Italian Treasury for public debt management and the software that implements these models, described in this paper, are the result of a long lasting effort to support the Italian Treasury in their decisions relating to issuance portfolios. The mathematical model of the public debt securities actually issued by the Italian government allows for fine details to be incorporated in the calculations, in order to faithfully reproduce the cash flows of the entire public debt portfolio. This means that the model can be used both as balance computational tool and as an expenditure forecasting tool in a unified framework of analysis.

At the core of the forecasting modules, there is the interest rates scenario generator. Extensive work has been carried out to assure its reliability and guarantee robustness and consistency of subsequent analysis both in terms of cost/risk frontier and CVA computations. The cost/risk analysis relies on a dynamic computation of various cost and risk measures in order to reduce the dependence of the results on a specific computational time window as much as possible.

While the work done to develop the tools described in this paper has benefitted from much academic literature on term structure modelling and credit risk measurement, it should be clear that no existing model in the literature can be used to address the complex issues that surround the debt management operations of the Italian Treasury. This means that several of the research hurdles that were encountered had no solution in existing published research. In general, for example, a government issuing different types of debt instruments (e.g. nominal, inflation-linked, foreign currency denominated) needs tools that allow the generation of scenarios for multiple term structures of interest rates for these instruments simultaneously. Apart from the obvious complexity of endogenously generating multiple term structures, there is a further need to satisfy regulatory or self-imposed constraints in the determination of feasible issuance portfolios, which makes standard cost/risk optimization tools inadequate. This need for multi-term-structure, constrained modelling contrasts with the common approach in published research that tends to analyze a single term structure of interest rates in isolation. We hope that the description of this model and of the general framework within which the Italian Treasury operates in terms of debt management will spur further research among scholars interested in this area which can, in turn, help finding more effective solutions. Indeed, since new circumstances and requirements manifest themselves at regular intervals in the world of public debt management, research and development remains ongoing in terms of models, metrics, and consideration of new types of financial instruments that the Italian Treasury expects to propose to the market.

## Appendix

### Generation of interest rate curves

In this appendix we present some results about the scenarios generator model of term structures of interest rates and inflation implemented in SAPE. As stated above, the objective of the stochastic model is the production of scenarios that can be used for conducting cost-risk analyses on feasible portfolios of issuances of sovereign bonds. In this regard the parameters of the vector autoregressive (VAR) model are checked regularly for stability and re-calibrated by using most recent market data. Clearly the models considered here are not to be considered as structural models based (solely) on deep parameters, rather as reduced-form characterizations of the unknown underlying process generating interest rates and inflation data. Hence, instability of the parameters is to be expected and re-calibration is required from time to time.

Hereafter, we describe an example of scenarios generation for the cost-risk analyses carried out by the Italian Department of the Treasury. It is worth noting that the data do not necessarily correspond to a distribution of scenarios actually used by the Italian Treasury.

In the example presented below, the stochastic scenarios of the yield curves are generated using the calibration of the model based on the historical sample of data from January 2005 to October 2019. The term structures considered are obtained considering: 12 maturities (from 3 months to 50 years) for the sovereign curve, 10 maturities (from 3 months to 30 years) for the European swap curve, 15 maturities (from 2 years to 30 years) for the European break-even inflation, and 11 maturities (from 3 months to 10 years) for the USD swap curve. This calibration shows excellent in-sample behavior, with very small pricing errors across the daily historical sample. As an illustrative example, in Figure 9 and in Figure 10 is shown the fit of the estimated yield curves for the observations of October 2006 and June 2016 (end of month):

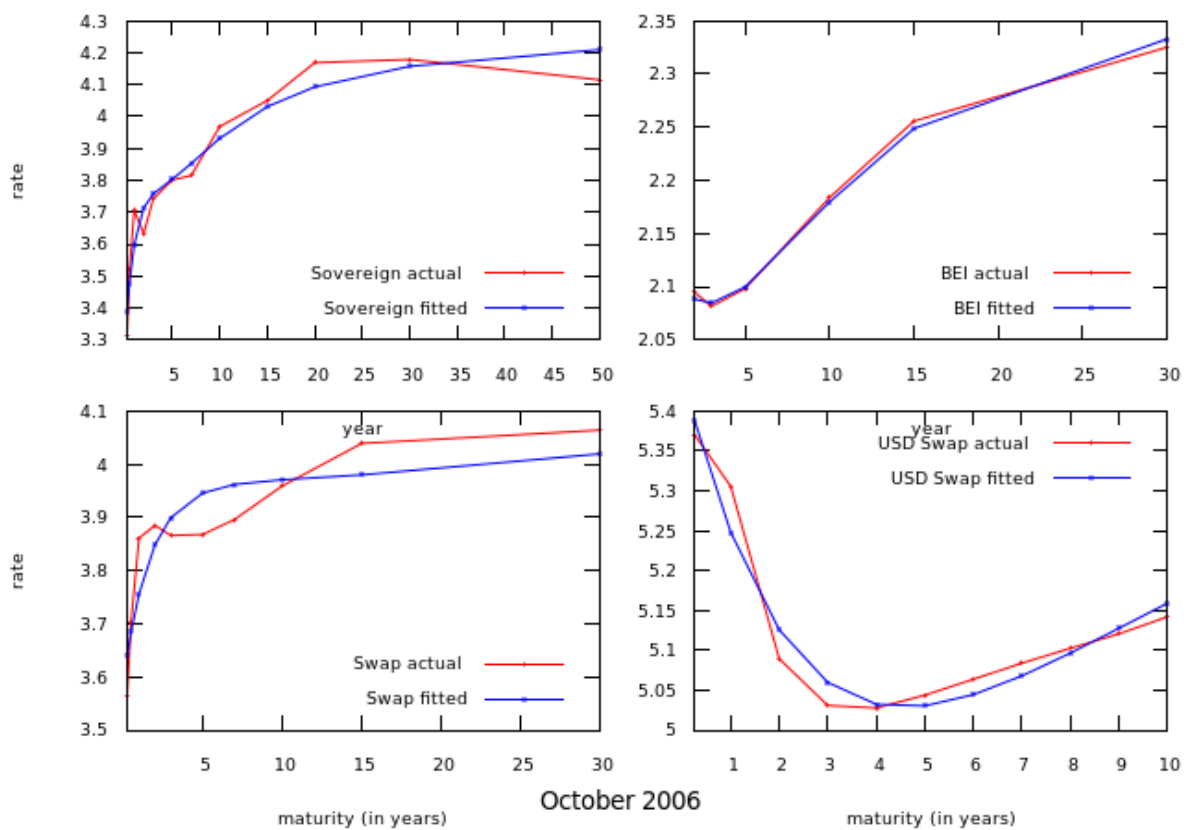


Figure 9 Actual vs Fitted in-sample yield curves

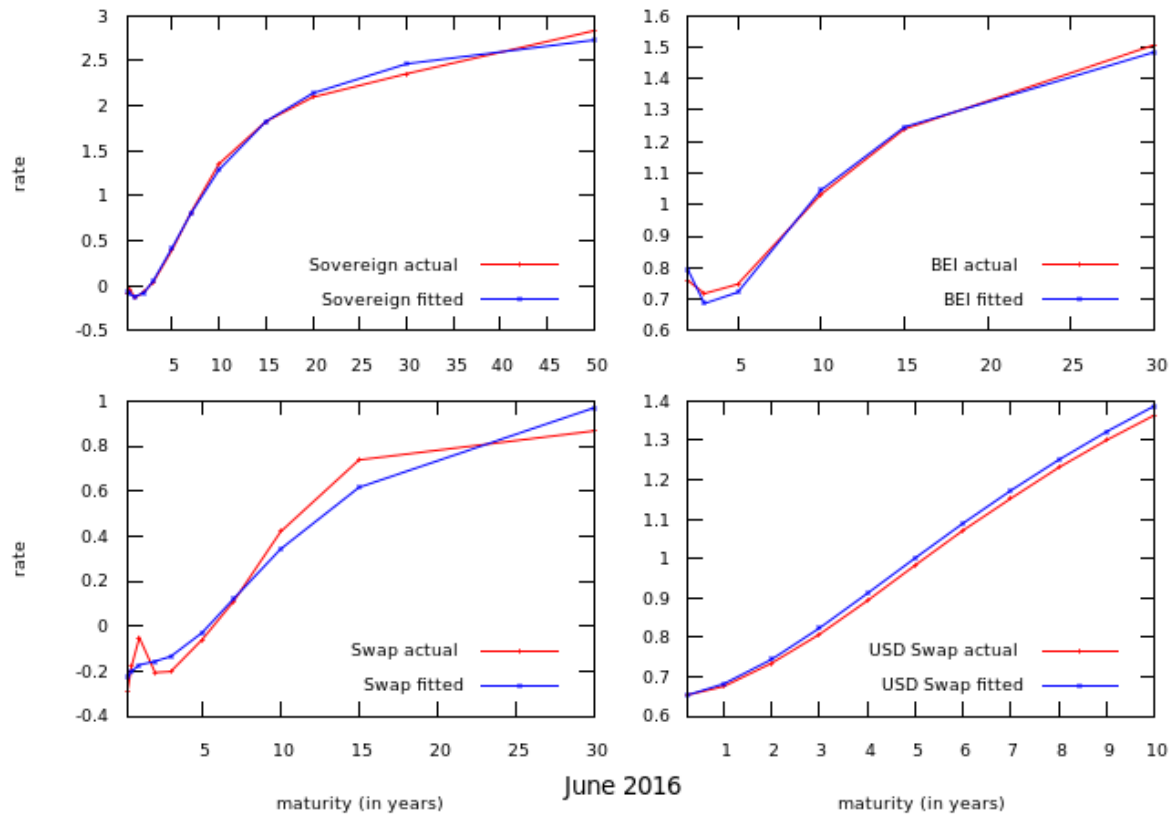


Figure 10 Actual vs Fitted in-sample yield curves

The time series of the  $\beta$  factors of the Nelson-Siegel-Svensson specifications, estimated in sample, are then projected out-of-sample through the generation of stochastic scenarios, obtained by bootstrapping the residual matrix of the VAR system. These scenarios are then used for the following cost-risk analyses on the portfolios of issuances, in combination with the liquidity and refinancing constraints of the Italian public debt structure. Overall, the procedure outlined above defines the stochastic component of the entire process of building the efficient cost-risk frontier on which the Treasury assesses its issuing policy for future periods.

For descriptive and interpretative purposes, it is interesting to analyze the out-of-sample behaviour of the yield curves, through a generation of 5,000 scenarios simulated from November 2019 to December 2025. If we consider a synthesis of the four different yield curves simulated by the model, Figure 11 shows the average out-of-sample behavior of the four term structures.

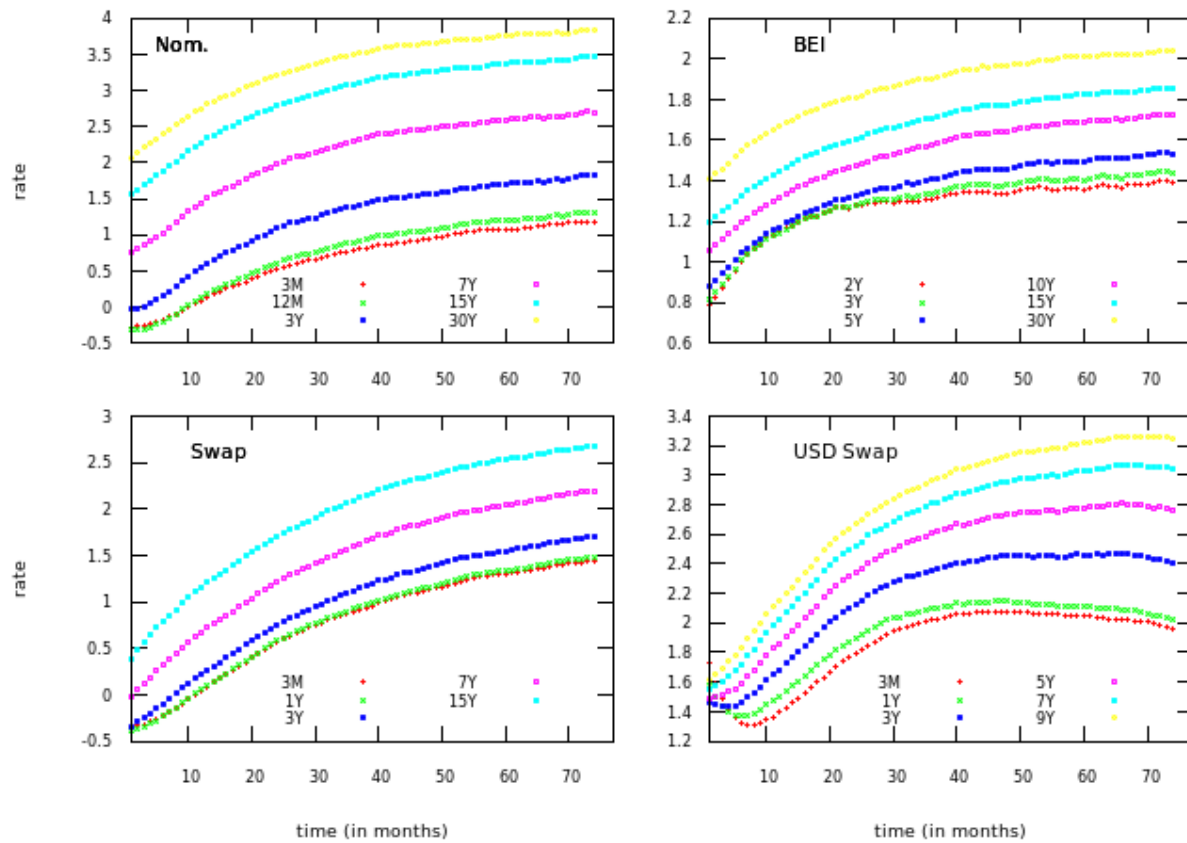


Figure 11 Average out-of-sample scenarios (Sovereign, BEI, Euroswap and USD swap time series)

It is also interesting to observe the behaviour of the curves in case of anchoring the short term sovereign rate (3 months BOT) to an exogenous prior. The anchoring module described in the main text is frequently used by the Italian Treasury for “what-if” analyses, granting the consistency with different hypotheses related to expectations in terms of issuance and monetary policy choices.

For example, assuming a prior of the 3-month rate equal to 0.1% in the first month of the out-of-sample simulation (November 2019), the average rotation (tilt) of the sovereign yield curve turns out to be as follows (see Figure 12):

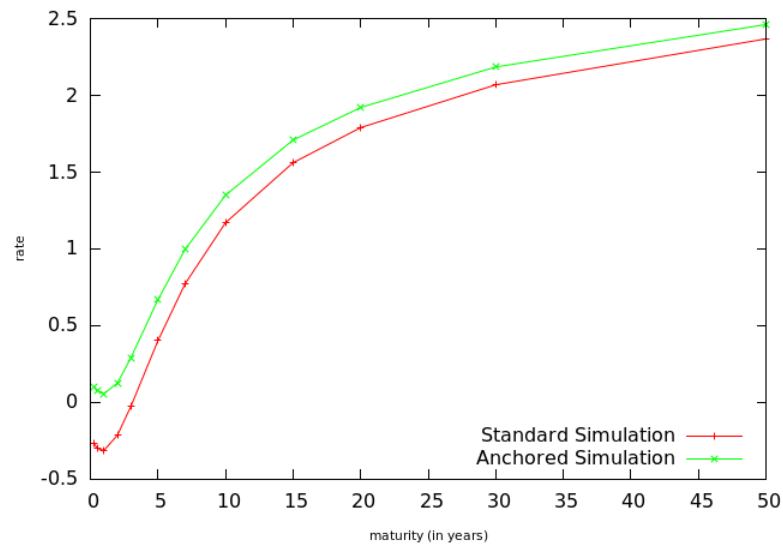


Figure 12 Anchoring the Sovereign yield curve.

Consistently, all the 5,000 simulated scenarios are tilted applying the monthly increments of the baseline simulation to the first, anchored, out-of-sample observation.

## References

- **C. Altavilla, R. Giacomini and G. Ragusa.** Anchoring the yield curve using survey expectations, CeMMAP working papers CWP52/13, *Centre for Microdata Methods and Practice, Institute for Fiscal Studies*, 2013.
- **E. Balibek and H. A. Memis.** Turkish treasury simulation model for debt strategy analysis. *The World Bank*, 2012.
- **T. Belton, K. Dawsey, D. Greenlaw, H. Li, S. Ramaswamy and B. Sack.** Optimizing the maturity structure of US Treasury debt: A model-based framework. *Hutchins Center Working Paper 46*, 2018.
- **M. Bernaschi, D. Vergni et al.,** Optimal Strategies for the Issuances of Public Debt Securities, *International Journal of Theoretical and Applied Finance*, 7(7) 2004.
- **M. Bernaschi, M. Briani, M. Papi and D. Vergni.** Scenario-Generation Methods for an Optimal Public Debt Strategy, *Quant. Finance* 7 (2) 2007.
- **J. H. Christensen, F. X. Diebold, and G. D. Rudebusch.** An Arbitrage-Free Generalized Nelson-Siegel Term Structure Model. *PIER Working Paper Archive*. Penn Institute for Economic Research, Department of Economics, University of Pennsylvania, 2008.
- **R.H. Clarida, L. Sarno, M.P. Taylor and G. Valente.** The Role of Asymmetries and Regime Shifts in the Term Structure of Interest Rates, *Journal of Business*, 79, 1193-1225, 2006.
- **L. Codogno, C. Favero and A. Missale.** Yield spreads on EMU government bonds. *Economic Policy* 18, 505-532, 2003.
- **P. Della Corte, L. Sarno and D. L. Thornton.** The Expectation Hypothesis of the Term Structure of Very Short-Term Rates: Statistical Tests and Economic Value, *Journal of Financial Economics*, 89, 158-174, 2008.
- **D. Dottori and M. Manna.** Strategy and tactics in public debt management, *Journal of Policy Modeling* 38.1, 1-25, 2016.
- **C. Favero and A. Missale.** Sovereign Spreads in the Euro Area. Which Prospects for a Eurobond? *CEPR Discussion paper* no. 8637, 2011.
- **F. X. Diebold and C. Li.** Forecasting the Term Structure of Government Bond Yields. *Journal of Econometrics*, 130(2), 337–364, 2006.
- **M. Gilli, S. Große and E. Schumann.** Calibrating the Nelson–Siegel–Svensson model, *Working Papers 031*, COMISEF, 2010.
- **R. S. Gürkaynak, B. Sack, and J. H. Wright.** The U.S. Treasury Yield Curve: 1961 to the Present. *Federal Reserve Board Finance and Economics Discussion Series*, 2006-28, 2006.
- **R. S. Gürkaynak, B. Sack and J. H. Wright.** The TIPS yield curve and inflation compensation. *Federal Reserve Board Finance and Economics Discussion Series*, 2008-05, 2008.
- **J. Hull and A. White,** One factor interest rate models and the valuation of interest rate derivative securities, *Journal of Financial and Quantitative Analysis*, 28 (2) 1993.
- **C. R. Nelson and A. F. Siegel.** Parsimonious Modeling of Yield Curves. *Journal of Business*, 60(4), 473–489, 1987.

- **L. Sarno, D. L. Thornton and G. Valente.** The Empirical Failure of the Expectations Hypothesis of the Term Structure of Bond Yields, *Journal of Financial and Quantitative Analysis*, 42, 81-100, 2007.
- **L. E. O. Svensson.** Estimating and Interpreting Forward Interest Rates: Sweden 1992–1994. *IMF Working Paper* 94/114, 1994.